Lec 1 Sep 7th

at the end of this course ask the q: Is the gradient a vector?

This course is useful in 2 aspects
1. General Relativity needs diff geo because of curved space and time
2. learn powerful and beautiful tools to describe physics in any geometry

Chapter 1 - on some basic Mathematics

we need some maths to be able to define a manifold (Thursday of next week)

1.1 $\mathbb{R}^n$ and its topology

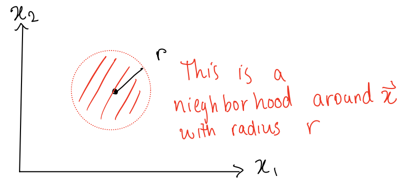
A point in $\mathbb{R}^n$ is an n-tuple $(x_1, \dots, x_n)$ w $\forall x_i \in \mathbb{R}$. The idea of continuous is that any 2 points in $\mathbb{R}^n$ have a line connecting them that exists in $\mathbb{R}^n$.

ex. Integers are not contiuous (discrete)

The continuity of a space defines its topology. Here we focus on local vs global topology. We use distance to define the topology.

Recall, the distance between $\vec{x}, \vec{y} \in \mathbb{R}^n$ is $d(x,y) = [(x_i - y_i)^2 + \dots + (x_n - y_n)^2]^{1/2}$

A neighborhood of radius r of $\vec{x} \in \mathbb{R}^n$ is the set of points s.t. $N_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n, d(\vec{x}, \vec{y}) < r\}$



A set of points in $\mathbb{R}^n$ are discrete if there exists a neighbourhood about each point that contains no other points.

A set of points $S \in \mathbb{R}^n$ is open $\forall x \in S$ $\exists$ a neighbourhood all in S

Example
1. $S = \{x \mid a < x < b\}$ is open
2. $S = \{x \mid a \leq x < b\}$ is not open. because $x = a$ does not have a neighbourhood all with in S

Note: Open sets cannot contain boundry points.

$\mathbb{R}^n$ has the hasdorff property, which means that any 2 points in $\mathbb{R}^n$ have neighbourhoods that do not intersect.

$d(\vec{x}, \vec{y})$ induces a topology on $\mathbb{R}^n$, which says that d determines whether a set is open or not.

Open sets have the following propeties:
1. empty set $\emptyset$ and the whole set S are open
2. If O_1 & O_2 are open sets the $O_1 \cap O_2$ is open
3. The union of open sets (finite number) is open

The topology of a set consits of the set and all the open sets in that set.

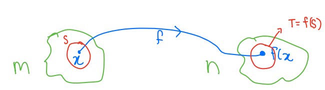
Any distance function induces the natural topology in $\mathbb{R}^n$

For example, $d'(\vec{x}, \vec{y}) = [4(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots (x_n - y_n)^2]^{1/2}$ has the same induced topology as any other distance fcn

You can define a topology without distance.

§1.2 Mappings

A map from M to N associates an elemet $x \in M$ to a unique $y \in N$



$f: M \to N$
$f: x \longmapsto f(x)$

S is a subset of M and the image of S under f is $f(S) = T$. The inverse Image of T is $f^{-1}(T) = S$

f can be many to one. If all points in $f(S)$ have a unique inverse in S then f is 1-1 and $\exists$ a one to one map f^{-1} Called the inverse of f

example: $\sin(x)$ is many ~~to~~ one b/c $\sin(x) = \sin(x+2\pi)$

Notation: $f: M \to N$ f maps M to N
$f: x \mapsto y$ f maps x to y

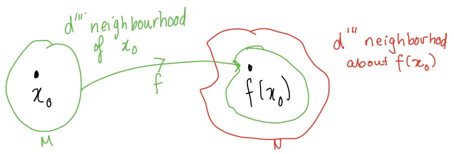
Given $f: M \to N$ and $g: N \to P$ then $\exists$ a composition map $g \circ f: M \to P$ such that $(g \circ f)(x) = g(f(x))$. If $f: M \to N$ then
f defined $\forall$ points in $M \Rightarrow$ f maps M into N
f defined $\forall$ points in $N \Rightarrow$ f maps M onto N
If f is both 1-1 and onto the f is a bijection. If f have an inverse, then f is 1-1

New [A map $f: M \to N$ is continuous at $x \in M$, if any open set in N containing $f(x)$ contains ~~the~~ image of an open set M contains x.

f is continuous on M if it is continuous $\forall x \in M$

old [Look at how this is related to continous defined in calculus. Recall, f is continuous at x_0 if $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t
$|f(x) - f(x_0)| < \varepsilon$ if $|x - x_0| < \delta$

We define $d'''(x, x_0) = |x - x_0|$ then our definition can be rewritten as follows: f is continuous at x_0 if $\forall$ d'''- neighbourhoods of $f(x_0)$ contains the image of a d''' neighbourhood of x_0



Theorem: $f: M \to N$ is continuous iff the inverse image of every open set is open in M

Lec 2 Sep 12

Offical Assignment will be released tonight. Crowdmark link will be sent out. AMATH 433?

Today's Topics: ① Real Analysis ② Group theory ③ Linear Algebra ④ Algebra of Square Matrices

§1.3 Real Analysis

$f(x)$ is analytic at $x = x_0$ if it has a Taylor expansion about x_0 with a non-zero radius of convergence

Analytic functions (C^ω) Which is a subset of C^∞

We will assume functions are analytic, but we'll often say smooth (C^∞)

An operator A of functions is a map that takes a function and yeilds another function
Examples: $A(f) = g(f)$, g is a function
$D(f) = \frac{df}{dx}$ where f is C'

The commutator of 2 operators A, B on f is $[A,B](f) = (AB - BA)f$ OR $A(B(f)) - B(A(f))$. If $[A,B]=0$ $\forall$ functions then A an B commute

Example $A = \frac{d}{dx}$ and $B = x\frac{d}{dx}$

$$[A,B](f) = \frac{d}{dx}\left(x\frac{df}{dx}\right) - x\frac{d}{dx}\left(\frac{df}{dx}\right)$$

$$= x\frac{d^2f}{dx^2} + \frac{df}{dx} - x\frac{d^2f}{dx^2} = \frac{df}{dx}$$

hence A & B do not commute

[function spaces dont have a good inntution on the commutator but in other contexts there is, such as Later in the course]

§1.4 Group theory

A set of elements G with a binary operation $\cdot$ is a group if

[Gi] Associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

[Gii] Identity: $\exists\, e \in G$ such that $x \cdot e = e \cdot x = x \quad \forall x \in G$

[Giii] Inverse: $\forall x \in G\ \exists\ x^{-1} \in G$ s.t. $x \cdot x^{-1} = x^{-1} \cdot x = e$

And is closed under the operation

A group is abelion (commutative) if
[Giv] $x \cdot y = y \cdot x$

Example: ① Set of permutation of n objects
② Rotations of a regular polygon

Aside: the Inverse is unique, and the identity is unique

A subgroup is simply a group that is contained within the group

Example: a set of permutation of n objects where the first element is unchanged.
This is (identical) similar to the permutations of $(n-1)$ objects [isomorphism]

§ 1.5 Linear Algebra

A set V is a vector space (over $\mathbb{R}$) if it has a binary operation $+$ where it is an abelian group and satisfies the following under multiplication. Let $\vec{x}, \vec{y} \in V$, $a, b \in \mathbb{R}$

[V_i] $a \cdot (\vec{x} + \vec{y}) = (a \cdot \vec{x}) + (a \cdot \vec{y})$
[V_{ii}] $(a+b) \cdot \vec{x} = (a \cdot \vec{x}) + (b \cdot \vec{x})$
[V_{iii}] $(ab) \cdot \vec{x} = a \cdot (b \cdot \vec{x})$
[V_{iv}] $1 \cdot \vec{x} = \vec{x}$

The identity under addition is $\vec{0} = 0$

Example
1. $n \times n$ matrices
2. Continuous real function on $a \le x \le b$

✡ Dual spaces become critical in the next 2 weeks ✡

Notation: we often drop $\cdot$ & ( ) and write $a\vec{x} + b\vec{y}$

A set $\{x_1, x_2, \dots, x_n\}$ is linerally dependent if $\exists \{a_1, \dots, a_n\}$ $a_i \neq 0$ s.t. $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n = 0$

if $\forall a_i = 0$ then the set is linearly independent

A vector space has a basis, which of the dimension of V, and allows us to generate any element of V

If $\vec{x}_i$, $i = 1, \dots, n$ is a basis of V then $\forall \vec{y} \in V$, $\exists a_i$'s such that

$$\vec{y} = \sum_{i=1}^{n} a_i x_i$$

A set of vectors $\{\vec{y}_1, \dots, \vec{y}_n\}$ generates a subspace of V with $a_1\vec{y}_1 + \dots + a_m\vec{y}_m$ $a_i \in \mathbb{R}$ $i = 1, \dots, m$

if $m < n \Rightarrow$ proper subspace

A normed vector space is one $\bar{w}$ a mapping from V to $\mathbb{R}$ s.t. $a \in \mathbb{R}$, $\vec{x}, \vec{y} \in V$

[N_i] $n(\vec{x}) \geq 0$ & $n(\vec{x}) = 0$ iff $\vec{x} = 0$

[N_{ii}] $n(a\vec{x}) = |a|\, n(\vec{x})$

[N_{iii}] $n(\vec{x} + \vec{y}) \leq n(\vec{x}) + n(\vec{y})$

Examples If $V = \mathbb{R}^n$ then

$n(x) = d(\vec{x}, 0) = [x_1^2 + \dots + x_n^2]^{1/2}$

$n'(\vec{x}) = d'(\vec{x}, 0) = [4x_1^2 + \dots + x_n^2]^{1/2}$

$n'''(\vec{x}) = d'''(\vec{x}, 0) = \max(|x_1|, \dots, |x_n|)$

All 3 satisfy N_i, N_{ii}, N_{iii}. In addition, some norms satisfy the parallelgram rule

[N_{iv}] $[n(\vec{x} + \vec{y})]^2 + [n(\vec{x} - \vec{y})]^2 = 2(n(\vec{x}))^2 + 2(n(\vec{y}))^2$

n, n' satisfy N_{iv} by n''' does not

If we have all 4 proprieties then we can define a bilinear symmetric inner product.

$$\vec{x}\cdot\vec{y} = \frac{1}{4}\left[n(x+y)\right]^2 - \frac{1}{4}\left[n(\vec{x}-\vec{y})\right]^2$$

bilinear : $(a\vec{x} + b\vec{y})\cdot z = a(\vec{x}\cdot\vec{z}) + b(\vec{y}\cdot z)$

$\vec{z}(a\vec{x} + b\vec{y}) = a(\vec{z}\cdot\vec{x}) + b(\vec{z}\cdot\vec{y})$

Symmetry : $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$

Positive definite $\vec{x}\cdot\vec{x} \geq 0$ and $\vec{x}\cdot\vec{x} = 0$ iff $\vec{x} = \vec{0}$

$n(\vec{x})$ on $\mathbb{R}^n$ is the Euclidean Norm, $\mathbb{R}^n$ with the Euclidean norm is denoted $\mathbb{E}^n$

A pseudonorm is a norm that violates N_i & N_{iii}, This occurs in special relativity What is the History of the Psuedo Norm

§1.6 Algebra of Square Matrices

A Linear transformation T on a vectorspace is a map from V onto V which is Linear

$$T(a\vec{x} + b\vec{y}) = a\,T(\vec{x}) + b\,T(\vec{y})$$

If $\{\vec{e}_i\}$ $i = 1,\dots,n$ is a basis for V, then $\vec{x} = \sum_{i=1}^{n} a_i\vec{e}_i$

and $T(\vec{x}) = T(\sum_i^n a_i\vec{e}_i) = \sum_i^n a_i T(\vec{e}_i)$ The $T(\vec{e}_i)$ can be expressed as $T_{ij}\vec{e}_j$

$= \sum_i^n a_i \sum_j^n T_{ij}\vec{e}_j$

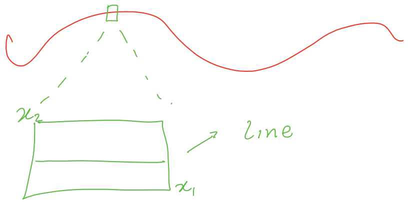
Where T_{ij} are the components of transformat T and are often written in matrix form.

If $\vec{A}, \vec{C}$ ar vectors and B is a matrix then $\vec{A}^T B\vec{C} = \sum_{ij}^{n} A_i B_{ij} C_j$ Strongly encourage to write in this form and not switch indices

Lec 3 Sep 14

Newtons law can be expressed without any coordinates. The manifold is loccally like $\mathbb{R}^n$

example:



§ 2.1 Definition of a Manifold

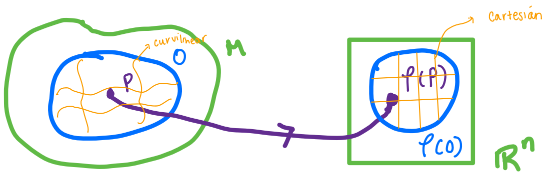
Idea: Any smooth curve / surface / volume [any dimension] looks locally like $\mathbb{R}^n$

A set of points M is a manifold if

$\forall x \in M$ has an open neighbourhood that has a continuous map which is 1-1 onto to map (bijective) open set of $\mathbb{R}^n$. then M has dimension n.

In this framework, there is no measure of length on M. Distance is a global proprety and we will discus this later.

Points in M look like $\mathbb{R}^n$, not $\mathcal{E}^n$, unless we have a metric



What is the difference between map, chart, atlas?

$\varphi: O \to \mathbb{R}^n \; [x^1, x^2, \dots x^n]$

O is an open set

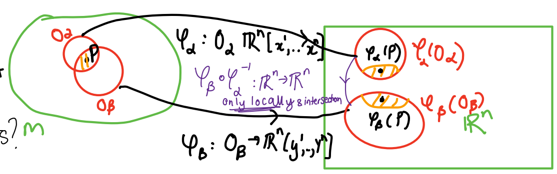
(O, φ) is a chart

$(x^1, x^2, \dots, x^n)$ are the coordinates of $\mathbb{R}^n$

There can be multiple charts at a given point on a manifold. These chart must overlap

Q: WTF is para-compact

Q: is pictures same w closed sets?



Q: if they maintain intersection, does that mean area & angle persved, is possible? $\Rightarrow$ metric

A collection of charts $(O_\alpha, \varphi_\alpha)$ is an atlas. This covers the manifold.

Defn continuous and bijective

Note: $\varphi_\alpha : O_\alpha \to \mathbb{R}^n$ & $\varphi_\beta : O_\beta \to \mathbb{R}^n$ are homeomorphic $\Rightarrow$ $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and is a coordine transfomation

$$y^i = y^i(x^1, \dots, x^n) \quad i = 1, \dots, n$$

We say φ_α φ_β are C^k related if all the partial derivatives of order k are continuous

If all $P \in M$, for all charts in M, is C^k-related then M is a C^k- manifold.

We assume M is a C^∞- manifold [differentiable]

Examples: $\mathbb{R}^n$ is a n- differentiable manifold

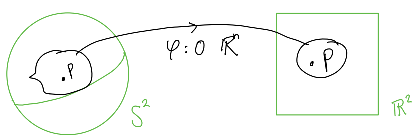
① $\mathbb{R}^n$ has the natural topology
② Can use identity map for the charts [any point in $\mathbb{R}^n$ maps to itself]

§2.2 The Sphere is a Manifold

The two-sphere in $\mathbb{R}^3$ is denoted by S^2 and defined by

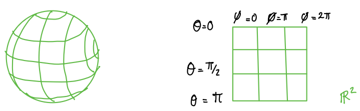
$$(x^1)^2 + (x^2)^2 + (x^3)^2 = \text{constant}$$

A one-sphere is a circle



We can map small neighbourhoods of P to a disc in $\mathbb{R}^2$. This map does not perserve length or angles

Another way to do this mapping is to use Spherical coordinates



$\theta = x^1 \qquad 0 \leq x^1 \leq \pi$ colatitude

$\phi = x^2 \qquad 0 \leq x^2 \leq 2\pi$ longitude

The map has problem at:

$\theta = 0, \pi$ the line is mapped to a point

$\phi = 0$, this line gets mapped to 2π

A solution restricted to $0 < x^1 < \pi \quad 0 < x^2 < 2\pi$ yields a chart for almost the entire sphere.

Another chart could be a similar system but where $\phi = 0$ at the equator and then go from $\phi = -\pi/2$ to $\phi = \frac{\pi}{2}$

Assignment 2 will have a Q on stereograpnic projects.

§2.3 Other examples of Manifolds

A set M that can be parameterized continuously is a manifold and its dimension is the number of indepedent parameters

① Set of rotations of a Rigid Object of 3D. Dimension 3 (Euler Angles)

② All (pure boosts) Lorentz transformations is a manifold of dimension 3.
The parameters are the components of Velocity

③ N particles in 3D, 3N dimensions for the position and 3N for Velocity. This is a manifold of dim 6N

④ An algebraic or differential equation for a dependent variable y in terms of an indep var x,
The set (y, x) is a manifold

⑥ A vector space V over $\mathbb{R}$ is a manifold. Suppose V is n-dim with basis $\{\vec{e}_1, \dots, \vec{e}_n\}$. Any $y \in V$ can be written as
$$\vec{y} = a^1\vec{e}_1 + \dots + a^n\vec{e}_n$$
We have a mapping $\vec{y} \to (a^1, .. a^n)$ & is from V to $\mathbb{R}^n$. It turns out that V is identical (isomorphic) to $\mathbb{R}^n$

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§2.5 curves

A map from a C^∞ manifold to another manifold N that is C^∞ and a bijection is a C^∞ diffeomorphism from M to N [How does this compare to a homeomorphism, a homeomorphism is continuous (C^0) and a bijection]

Diffeomorphism's in a specialized case of a homeomorphism [a subset / subspace?]

A differentiable manifold is a set M such that all points in M have an open set that has a map (diffeomorphism) to an open set in $\mathbb{R}^n$, we say it has dimension N

A curve is a differentiable map, say γ (gamma), from an open set of $\mathbb{R}$ into M.

$$\gamma : [a, b] \to M \quad \text{or} \quad \gamma \mapsto \gamma(\lambda) \in M$$

We parameterize the curve with lambda λ.

Two curves with the same image but different parameterizations are different.

Quote: Thank you for paying attention more so than me

Suppose the image of the curve is in the open set O with chart φ:

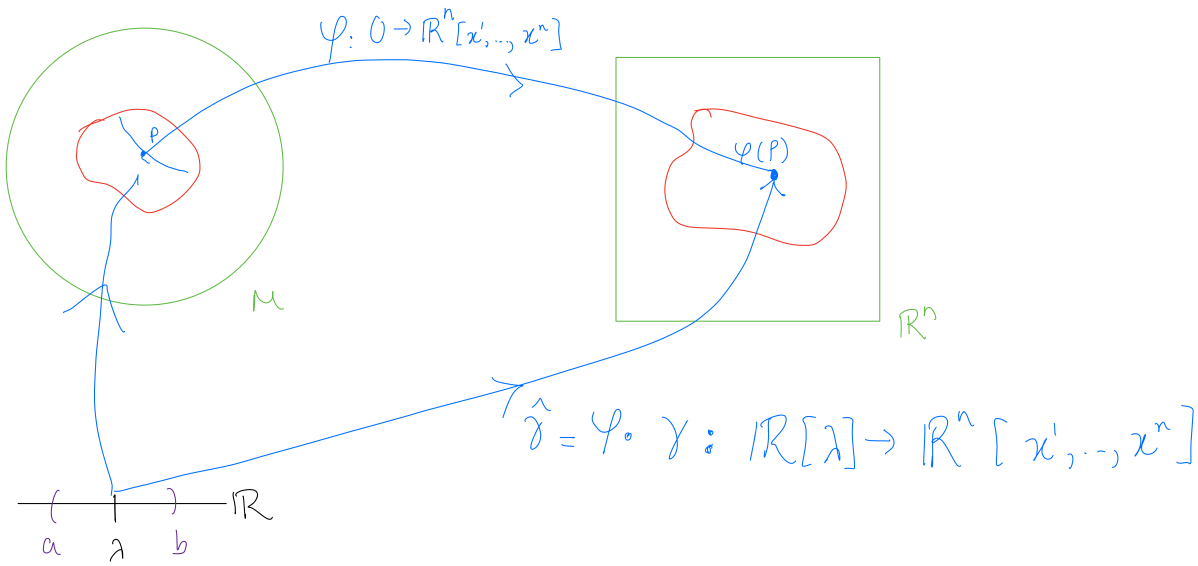
$$\varphi : O \to \mathbb{R}^n [x^1, \dots, x^n]$$

We obtain a coordinate representation of the curve:

$$\hat{\gamma} = \varphi \circ \gamma : \mathbb{R}[\lambda] \to \mathbb{R}^n [x^1, \dots, x^n]$$

OR

$$\lambda \mapsto (x^1(\lambda), \dots x^2(\lambda)) \equiv [x^1(\gamma(\lambda)), \dots, x^n(\gamma(\lambda))]$$



γ is differentiable if M is a C^∞ manifold

§ 2.6 Functions

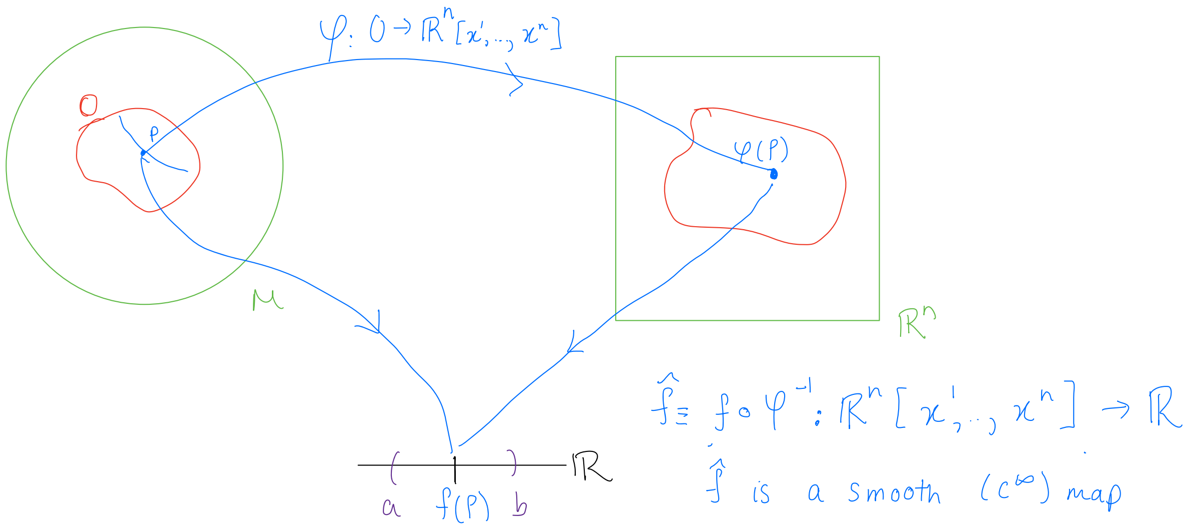
A Function, say f, on M is a smooth map from M to $\mathbb{R}$,

$$f: M \to \mathbb{R} \quad \text{or} \quad x \longmapsto f(x) \in \mathbb{R}$$

With chart $\varphi: O \to \mathbb{R}^n[x^1, \dots, x^n]$ we get a coordinate representation of f:

$$\hat{f} = f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\text{OR} \quad (x^1, \dots, x^n) \longmapsto \hat{f}(x^1, \dots, x^n)$$



On a manifold we always have coordinates but we don't always mention them explicitly

§ 2.7 Vectors and Vector Feild

In a manifold, no magnitutde for a vector.

Vectors typically have a direction and magnitude. In our definition, we will have a direction but no magnitude because we only have a local descption about each point.

Suppose we have a curve, γ, that passes through the point $P \in M$ with coordinates $\{x^i\}$ $i = 1, \dots, n$ and also a smooth function for M

$$\gamma: [a, b] \to M$$
$$f: M \to \mathbb{R}$$

We can evaluate f on the curve

$$g \equiv f \circ \gamma \equiv f \circ \varphi^{-1} \circ \varphi \circ \gamma = \hat{f} \circ \gamma : [a, b] \to \mathbb{R}$$

OR

$$\lambda \longmapsto f(x^1(\lambda), \dots, x^n(\lambda))$$

OR

$$f(x^i(\lambda)) : [a, b] \to \mathbb{R}$$

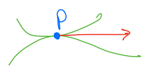
Since f and γ are both differentiable, so is g. We can differentiate g w.r.t x using the chain rule

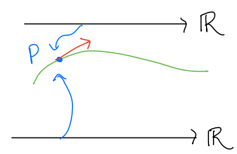
This is a directional derivative. What is a directional derivative?

$$\boxed{\frac{dg}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x_i}} \quad \Rightarrow \quad \left. \frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i} \right]$$ This is an operator

Observe that the above is a directional derivative of f in the direction of $\left\{\frac{dx^i}{d\lambda}\right\}$. These are the components of the tangent to the curve

Note that each tangent vector $\left\{\frac{dx^i}{d\lambda}\right\}$ has an infinite number an infinite number o curves that are tangent to it.

a) 

b) 

Example:

At the point $P = x^i(0)$, consider the curve

$$\hat{\gamma}_1 = (\varphi \circ \gamma_1)(\lambda) = \lambda a^i \quad [a^i \text{ constants}]$$

The Tangent is

$$\left. \frac{dx^i}{d\lambda} \right|_{\lambda=0} = a_i$$

Now consider a different curve at P

$$\hat{\gamma}_2 = (\varphi \circ \gamma_2)(\mu) = \mu^2 b^i + \mu a^i = x^i, \quad a^i\, b^i \text{ constants}$$

The tangent vector is $\left. \frac{dx^i}{d\mu} \right|_{\mu=0} = 2\mu b^i + a^i = a^i$

The curves γ_1 and γ_2 have the same tangent vector at P

It can be shown that tangents to the curve form a vector space. We can show all the properties of a vector space are satisfied but we only show closure.

<u>Proof</u>: Suppose $a, b \in \mathbb{R}$ and we have curves $\gamma_1(\lambda)$ and $\gamma_2(\mu)$

From γ_1: $\frac{d}{d\lambda} = \sum_{i=1}^{n} \frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}$

From γ_2: $\frac{d}{d\mu} = \sum_{i=1}^{n} \frac{dx^i}{d\mu}\frac{\partial}{\partial x^i}$

Consider the linear Superposition of the two,

$$a\frac{d}{d\lambda} + b\frac{d}{d\mu} = \sum_{i=1}^{n}\left(a\frac{dx^i}{d\lambda} + \frac{b\,dx^i}{d\mu}\right)\frac{\partial}{\partial x^i}$$

we introduce a new parameter $\emptyset$ such that

$$\frac{d}{d\emptyset} = \sum_{i=1}^{n}\left(a\frac{dx^i}{d\lambda} + b\frac{dx^i}{d\lambda}\right)\frac{\partial}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial x^i}{\partial\emptyset}\frac{\partial}{\partial x^i}$$

This proves closure, which means the Scalar sum of 2 tangent vectors is also a tangent vector ■

If you consider the tangent vectors along the coordinate lines, $\{x^i\}$, we get $\left\{\frac{\partial}{\partial x_i}\right\}$
This forms a basis to the vector space

In our equations for $\frac{d}{d\lambda}$, $\left\{\frac{dx^i}{d\lambda}\right\}$ are the components in the vector space.

Lec 5- Sep 21

All future assignments will be due on Thursdays @ 5^{pm}

Previously, we define $g = f\circ\gamma : (a,b) \to \mathbb{R}$, $\frac{dg}{d\lambda} = \sum_{i=1}^{n}\frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i} \Rightarrow \frac{d}{d\lambda} = \sum_{i=1}^{n}\frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}$ where $\frac{dx^i}{d\lambda}$ are the components of the tangent to the curve

With Coordinates $\{x^i\}$, we get coordinate lines $\left\{\frac{\partial}{\partial x^i}\right\}$ which form a basis.

Since there is a 1 to 1 correspondnce between the tangent vectors at P and the space of partial derivatives at P, we use $\frac{\partial}{\partial x^i}$ to denote the tangent vectors to the curve

§ 2.8 Basis vectors and basis vector fields

Every point P in manifold M has a tangent space denoted by T_p or T_pM, which is a vectorspace, w̄ same dim (
dim(M) = n.

We need n linearly independent vectors in T_pM to form a basis. A coordinate system $\{x_i\}$ at P has a coordinate basis of $\left\{\frac{\partial}{\partial x^i}\right\}$ of T_pM for all $P\in M$

Suppose $\{\bar{e}_i\}$ $i=1,\dots,n$ is another basis of T_pM. Any-vector in T_pM, say $\bar{V}$, can be written as

$$\bar{V} = \sum_{i=1}^{n} V^i\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} V^{j'}e_j$$

$\{V^i\}$ & $\{V^{j'}\}$ are the coordinates w.r.t. $\left\{\frac{\partial}{\partial x^i}\right\}$ and $\{\bar{e}_j\}$ These coordinates are functions on M.

Note: In $\left\{\frac{\partial}{\partial x^i}\right\}$ i is a superscript but appears in the denominator & ∴ is considered a subscript

For Vectors we use subscripts for the basis and superscripts for coordinates

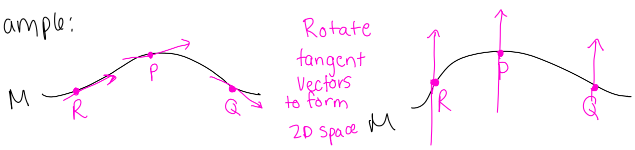
A vector is an object that lives in T_pM. A vector field are mappings that defines a vector for all $P\in M$

A vector field is diff'able if its coordinates are differentiable. The basis $\{\frac{\partial}{\partial x^i}\}$ are linearly indepent if $\{x^i\}$ are (proper) coordinates.

§2.9 Fiber Bundles

A manifold M with a tangent space T_pM can be combined to form a tangent Bunde (TM)

1D example:



The tangent Bunde is a 2D space. (in general 2n) is also a manifold. We can define a projection to get the point from the tangent bundle. The tangent bundle is an example of a fiber bundle.

§2.12 Vector Feilds and Integral Curves

Any curve on a manifold has a tangent vector at every point. Since this is true $\forall$ points on the curve this defines a vector field. It can be shown that any smooth vector field has a curve associated with it this is an Integral Curve.

Vector fields coorespond to a system of first order ODEs and the integral curve is the solution.

Supose we have a smooth vector field $\bar{V} \in T_pM$ with components $V^i(P)$ with coordinates $\{x^i\}$ Then if we write $V^i(P) = v^i(x^1, \ldots, x^n)$. Then we get a system of DEs

$$\boxed{\frac{dx^i}{d\lambda} = v^i(x^1, \ldots, x^n)}$$

If V^i is C^1 $\forall i$ then $\exists$ a soln to the system which is the Integral Curves.

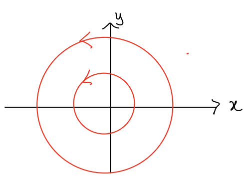
Example:

$$\bar{V} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\frac{dx}{d\lambda} = -y \qquad \frac{dy}{d\lambda} = x$$

If you differentiate the first

$$\frac{d^2x}{d\lambda^2} = -\frac{dy}{d\lambda} = -x \qquad \text{Harmonic Oscillator}$$



§2.13 Exponentiation of the Operator $\frac{d}{d\lambda}$

Suppose we have an analytic (smooth) manifold (C^∞) with coordinates $\{x^i(\lambda)\}$ along Integral Curves

Then $\bar{Y} = \frac{d}{d\lambda}$ are analytic functions of λ and we can $\therefore$ Taylor Expand. We Taylor Expand $\{x^i\}$ about λ_0,

$$x^i(\lambda_0 + \epsilon) = x^i(\lambda_0) + \epsilon \left(\frac{dx^i}{d\lambda}\right)\Big|_{\lambda_0} + \frac{1}{2}\epsilon^2 \left(\frac{d^2 x^i}{dx^2}\right)\Big|_{\lambda_0} + \dots$$

$$= \left(1 + \epsilon \frac{d}{d\lambda} + \frac{\epsilon^2}{2}\frac{d^2}{d\lambda} + \dots\right) x^i \Big|_{\lambda_0}$$

$$\equiv \exp\left[\epsilon \frac{d}{d\lambda}\right] x^i \Big|_{\lambda_0} \quad \text{[THIS IS NOTATION]}$$

We have the exponentiation of the operator, which is short hand for the above expression.

Note: $\exp\left(\epsilon \frac{d}{d\lambda}\right) = e^{\epsilon \frac{d}{d\lambda}} = e^{\epsilon \bar{Y}}$ This is trying to give us a sense of distance

§ Lie brackets and non coordinate basis

Suppose $\{x^i\}$ is a coordinate system and $\{\frac{\partial}{\partial x^i}\}$ is a basis of vector fields

We know that any n linearly independent vectors form a basis but can a basis form a coordinate system? No

By construction $\frac{\partial}{\partial x^i}$ & $\frac{\partial}{\partial x^j}$ commute for all i, j

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0$$

Suppose we have $\bar{V} = \frac{d}{d\lambda}$ and $\bar{W} = \frac{d}{d\mu}$ we can show that they need not always commute

$$\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = \frac{d}{d\lambda}\left(\frac{d}{d\mu}\right) - \frac{d}{d\mu}\left(\frac{d}{d\lambda}\right)$$

$$= \left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\right)\left(\sum_{j=1}^n W^j \frac{\partial}{\partial x^j}\right) - \left(\sum_{j=1}^n W^j \frac{\partial}{\partial x^j}\right)\left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\right)$$

$$= \sum_{i,j} \left\{ V^i \frac{\partial W^j}{\partial x^i}\frac{\partial}{\partial x^j} + V^i W^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial x^i} - W^j V^i \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} \right\}$$

$$\boxed{\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = \sum_{i,j} V^i \frac{\partial W^j}{\partial x^i}\frac{\partial}{\partial x^j} - W^j \frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial x^i}}$$

If this is non zero then $\frac{d}{d\lambda}$ & $\frac{d}{d\mu}$ form a non-coordinate basis

The Lie braket of $\bar{V} = \frac{d}{d\lambda}$ and $\bar{W} = \frac{d}{d\mu}$ is $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right]$

Lec 6 - Sep 26

Q 1] just show they are inverses

A1 marks very latest next thursday

Q4) using C^∞ argue that commutator is 0

§ 2.14 Lie brackets & non-coordinate basis

The Lie bracket is defined as the comutator of two vectors: $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right]$

Geometric Interpretation:

Consider a coordinate basis $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}$ where $\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\right] = 0$

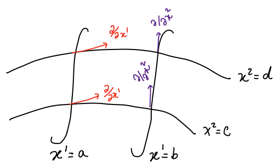
The Integral curve of $\frac{\partial}{\partial x^1}$ [that are tangent to $\frac{\partial}{\partial x^1}$]

$\frac{dx^1}{d\lambda} = 1$ and $\frac{dx^2}{d\lambda} = 0$

$\Rightarrow$ $x^1 = \lambda + c$ and $x^2 = \text{constant}.$

Integral Curves of $\frac{\partial}{\partial x^2}$ are $x^1 = \text{constant}$, $x^2 = \lambda + \text{constant}$

HOW DOES THIS RELATE TO DYNAMIC SYSTEMS and Vector fields discussed in Calc4

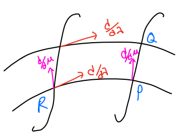


Locally things can be curvy but on a cure only 1 parameter changes

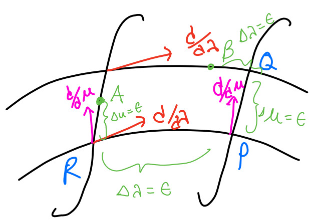
Along each integral curve all the x^i's are constant except for one that changes

Next consider a non-coordinate basis $\bar{V} = \frac{d}{d\lambda}$, $\bar{W} = \frac{d}{d\mu}$ with $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] \neq 0$

On the Integral Curves of $\frac{d}{d\lambda}$, λ increases and μ can also change.



Suppose we start a P and move $\Delta\lambda = \epsilon$ and then $\Delta\mu = \epsilon$ to end up at A



We can also start at P, move $\Delta\mu = \epsilon$ then $\Delta\lambda = \epsilon$ and end up at B

Find the approximate distance between A + B:

1^{st} Path, first move to R: $x^i(R) = \exp\left[\epsilon \frac{d}{d\lambda}\right] x^i \Big|_P$

Then to A,

$$x^i(A) = \exp\left[\epsilon \frac{d}{d\mu}\right] \exp\left[\epsilon \frac{d}{d\lambda}\right] x^i \Big|_P$$

Similarily, we move from P to Q to B

$$x^i(B) = \exp\left[\epsilon \frac{d}{d\lambda}\right] \exp\left[\epsilon \frac{d}{d\mu}\right] x^i \Big|_P$$

The difference between the two is

$$x^i(B) - x^i(A) = \left[\exp\left(\epsilon \frac{d}{d\lambda}\right), \exp\left(\epsilon \frac{d}{d\mu}\right)\right] x^i \Big|_P$$

$$= \ldots \quad \text{[Assignment 2 4(b)?]}$$

$$= \epsilon^2 \left[d/d\lambda, \frac{d}{d\mu} \right] + O(\epsilon^3)$$

$\bar{V}, \bar{W}$ are in a coordinate basis iff $[\bar{V}, \bar{W}] = 0$ $\quad \bar{V}$ and $\bar{W}$ are vector fields

§2.16 One - forms [co vectors]

T_pM is the space of tangent Vectors at $P \in M$. A one-form is a Linear, real valued function of Vectors

$$\tilde{\omega} : V \longrightarrow a \in \mathbb{R}$$

The Space of one forms is the dual space to the tangent space T_pM

Suppose $\tilde{\omega}$ is a one form and $\bar{V}$ is a vector, both at P, then we have an operation

$$\boxed{\tilde{\omega}(\bar{V}) \in \mathbb{R}}$$

One forms are linear with ($a, b \in \mathbb{R}$, $\tilde{\sigma}$ is a one form)

1. $\tilde{\omega}(a\bar{V} + b\bar{W}) = a\tilde{\omega}(V) + b\,\tilde{\omega}(\bar{W})$
2. $(a\tilde{\omega})(\bar{V}) = a(\tilde{\omega}(V))$
3. $(\tilde{\omega} + \tilde{\sigma})(V) = \tilde{\omega}(V) + \tilde{\sigma}(\bar{V})$

These proprieties ensure that one forms at P forms a vector space. This is called the dual space of T_pM called T_p^*M

Vectors are Linear, real-valued functions of one forms and hence T_pM is the dual of T_p^*M

$$T_p^{**}M = T_pM \qquad \text{[this is always the case]}$$

Example: $(a\tilde{\omega} + b\tilde{\sigma})(\bar{v}) = (a\tilde{\omega})(\bar{v}) + (b\tilde{\sigma})(\bar{v})$
$= a(\tilde{\omega}(\bar{v})) + b(\tilde{\sigma}(\bar{v}))$

Notation: $\tilde{\omega}(\bar{v})$ or $\bar{V}(\tilde{\omega}) = \langle\tilde{\omega}, \bar{V}\rangle$ these are all called contraction

coordinates have superscripts

Vectors are sometimes called Contravariant; One-forms (covectors) are covariant → coordinates have subscripts

§2.17 Examples of One-forms

example matrix algebra
colomn vectors are vectors
row vectors are one-forms

$(a,b) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto (a,b)\begin{pmatrix} x \\ y \end{pmatrix} = ax + by \in \mathbb{R}$

§2.18 dirac delta function

C^∞ functions are an abelian group under addition and a Vector Space under multiplication.

The dual space of the functions are one forms and called *distributions*

$$\delta(x) : f(x) \longmapsto \int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

§2.19 The gradient and the pictorial representation of a one-form

A vector field has a unique vector at every point. A field of one-form has a unique one form at every point.

Differentiabily of One forms will be determined in terms of diff'abitly of Vectors and functions

A Tangent Bundle, TM contains $M \times T_pM$. A cotangent Bundle contains T^*M contains $M \times T_p^*$
Both are fiber bundles

a function

We will show that the gradient of f, denoted $\tilde{d}f$ is a one-form anddefined as:

$$\boxed{\tilde{d}f\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} = \sum_{n=1}^{i} \frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i}} \in \mathbb{R}$$

gradient "exists" in Dual space and takes in from the Tangent Space

$\tilde{d}f$ is an element of $T_P^* M$ and the contraction with $\frac{dx^i}{d\lambda}$ yields the directional derivative of f a lo
a curve tangent to $\bar{V}$

Check $\tilde{d}f$ is a one form:

$$\tilde{d}f\left(a\frac{d}{d\lambda} + b\frac{d}{d\mu}\right) = \left(a\frac{d}{d\lambda} + b\frac{d}{d\mu}\right)f \quad \text{by above definition}$$

$$= a\frac{df}{d\lambda} + b\frac{df}{d\mu} = a\,\tilde{d}f\left(\frac{d}{d\lambda}\right) + b\,\tilde{d}f\left(\frac{d}{d\mu}\right)$$

we see $\tilde{d}f$ is a linear operator on vectors

$\frac{df}{d\lambda}$ at P is computed from $\frac{\partial f}{\partial x^i}$ at P and this forms the components of $\tilde{d}f$

Note $\frac{\partial f}{\partial x^i}$ has a lower index, opposite to be basis of vectors

Lec 7 - Sep 28th

§ 2.21 Basis 1-forms and components of 1-forms

Any n linearly independent one-forms are a basis of T_p^*M [cotangent space]

Given a basis of T_pM say $\{\bar{e}_i, i=1,\dots,n\}$ this induces a dual basis to T_p^*M $\{\tilde{\omega}^i, i=1\dots n\}$

If $\bar{V} \in T_pM$ then the dual basis $\tilde{\omega}^i$ is defined by

$$\boxed{\tilde{\omega}^i(\bar{V}) = V^i} \qquad \bar{V} = \sum_{i=1}^{n} V^i \bar{e}_i$$

and

$$\boxed{\tilde{\omega}^i(\bar{e}_j) = \delta^i_j \text{ [Kroneker delta]}}$$

We show that $\{\tilde{\omega}^j\}$ are linearlly undepent and form a basis of T_p^*M.

Consider any one-form $\tilde{q}$

$$\tilde{q}(\bar{V}) = \tilde{q}\left(\sum_{j=1}^{n} V^j \bar{e}_j\right)$$

Linear

$$= \sum_{j=1}^{n} V^j \tilde{q}(\bar{e}_j)$$

Define $q_j = \tilde{q}(\bar{e}_j)$ to be the components of $\tilde{q}$ on the dual basis to $\{\bar{e}_j\}$

Also $\tilde{q}(\bar{V}) = \sum_j q_j \tilde{\omega}^j(\bar{V}) \Rightarrow \tilde{q} = \sum_j q_j \tilde{\omega}^j$

Note: compare with $\bar{V} = \sum_j V^j \bar{e}_j$

Therefore, $\{\tilde{\omega}_j\}$ are a basis since there are n of them and we can generate any $\tilde{q}$ with This decomposition

It follows,

$$\boxed{\tilde{q}(\bar{V}) = \sum_j q_j V^j}$$

This is a Contraction

If $\{\bar{e}_i\}$ is a basis of T_pM $\forall$ points $U \subset M$ then $\{\tilde{\omega}_j\}$ is a basis T_p^*M $\forall$ points $U \subset M$

The coordinate basis $\{x^i\}$ on $\mathcal{U}$ defines a natural vector field $\{\partial/\partial x^i\}$ (a bais of T_pM)

and this defines a natural basis of one forms $\{\tilde{d}x^i\}$ [at each point]

With this notation, $\tilde{d}x^i(\partial/\partial x^j) = \delta^i_j$

§ 2.21 Index Notation

Components of vectors V^i

Components of one-forms ω_j

Vector basis: $\bar{e}_i$

One form basis $\tilde{\omega}^j$

Coordinate basis; (1-forms) $\tilde{d}x^j$

(Vectors) $\partial/\partial x^i$

Example : $\tilde{\omega}(\bar{V}) = \sum_j V^j \omega_j = V^j \omega_j$ [einstein's Summation Notation]

An index that occurs twice is summed over if one is a subscript and on is a superscript

Example: $\tilde{\omega} = \sum_j \omega_j \tilde{d}x^j \rightarrow \omega_j \tilde{d}x^j$

Examples with no sum: $V^j V^k$ not repeated index no sub; $V^j \omega_i$ no repeated index; $V^j W^j$ no subscripts

§ 2.22 Tensor and Tensor fields

We build on vectors and 1-forms to get tensors

At $P \in M$ a tensor of Type $\binom{N}{N'}$ is a Linear Map Operator that takes N- 1 forms and N′ vectors and Yeilds a real number.

Example: F is a $\binom{2}{2}$ tensor

We can write this as:

$$F(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{w})$$

Since it is Linear in all arguments

$$F(a\tilde{\omega} + b\tilde{\lambda}, \tilde{\sigma}; \bar{V}, \bar{\omega})$$
$$= a\, F(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{\omega}) + b\, F(\tilde{\lambda}, \tilde{\sigma}; \bar{V}, \bar{W})$$

$$F(\tilde{\omega}, \tilde{\sigma}; a\bar{V} + b\bar{U}, \bar{W}) = aF(\tilde{\omega}, \tilde{\sigma}; \bar{V}, \bar{W}) + b\, F(\tilde{\omega}, \tilde{\sigma}; \bar{U}, \bar{w})$$

§ 2.23 Examples of Tensors

In Linear Algebra, column vectors are vectors are $\binom{1}{0}$ tensors, Row "Vectors" are One-forms or $\binom{0}{1}$ tensors, and Matrix is a $\binom{1}{1}$ tensor

§ 2.24 Components of Tensors and the outer product

Consider 2 Vectors $\bar{V}, \bar{W}$. We can form a $\binom{2}{0}$ Tensor with the Outer product.

Chosen Variable

$$\bar{V} \otimes \bar{W} (\tilde{p}, \tilde{q}) = a \in \mathbb{R}$$
$$\equiv \bar{V}(\tilde{p})\, \bar{W}(\tilde{q})$$

Outer product
direct product
tensor product

If $\tilde{p}, \tilde{q}$ are one forms we can form a $\binom{0}{2}$ tensor

$$\tilde{p} \otimes \tilde{q} (\bar{V}, \bar{W}) = \tilde{p}(\bar{V})\, \tilde{q}(\bar{W})$$

chosen variable

$\otimes$ is the outer/direct/tensor product

The outer product of an $\binom{N}{M}$ tensor and an $\binom{N'}{M'}$ tensor is a tensor of order $\binom{N+N'}{M+M'}$

The components of a tensor are the values it takes when it has basis vectors and 1-form as arguments

Example: If S is a $\binom{3}{2}$ tensor, then on the basis $\{\bar{e}_i\}$ and $\{\tilde{\omega}_i\}$ has components

$$S^{ijk}{}_{lm} \equiv S(\tilde{\omega}^i, \tilde{\omega}^j, \tilde{\omega}^k; \bar{e}_l, \bar{e}_m)$$

Lec 8 Oct 3

§ 2.25 Contractions

$\bar{V} \otimes \tilde{\omega}$ is a $\binom{1}{1}$ tensor and is written as $V^i \omega_j$.

Consider examples with

S^i_{jk} is a $\binom{1}{2}$ tensor

$P^{\ell m}$ as a $\binom{2}{0}$ tensor

These can be contracted in varius ways

$\underbrace{S^i_{jk} P^{jm}}_{\text{contracted}}$ is a $\binom{2}{1}$ tensor

$S^i_{jk} P^{ks}$ is a $\binom{2}{1}$ tensor

In general, the 2 above differ [unless P is symetric]

Proprety: Contractions are independent of the basis

IDEA of Proof:

1. $\tilde{q}(\bar{V}) = \dots = q_i V^i$ see assignment 2 for the details
2. Consider A a $\binom{2}{0}$ tensor and B a $\binom{0}{2}$ tensor. A contraction of A and B is $A^{ij} B_{jk}$ $= C^i_k$ where C is a $\binom{1}{1}$ tensor

Consider C applied to $\tilde{\sigma}$ and $\bar{V}$, $C(\tilde{\sigma};\bar{V}) = C^i_k \sigma_i V^k$

$= A^{ij} B_{jk} \sigma_i V^k$

$= \sigma_i A^{ij} B_{jk} V^k$ components to operators?

$= \sigma_i A(\tilde{\omega}^i;\tilde{\omega}^j)\, B(\bar{e}_j,\bar{e}_k)\, V^k$

using tensors are linear

$= A(\sigma_i \tilde{\omega}^i;\tilde{\omega}^j)\, B(\bar{e}_j, V^k \bar{e}_k)$

$= A(\tilde{\sigma},\tilde{\omega}^j)\, B(\bar{e}_j,\bar{V})$

$= A(\tilde{\sigma}, B(\bar{e}_j,\bar{V})\tilde{\omega}^j)$

Aside: $B(\bar{e}_j,\bar{V})\tilde{\omega}^j = B_{\ell m}\tilde{\omega}^\ell \tilde{\omega}^m \bar{e}_j V^n \bar{e}_n \tilde{\omega}^j$

$= B_{\ell m} \underbrace{(\tilde{\omega}^\ell \bar{e}_j)}_{= \delta^\ell_j} \underbrace{(\tilde{\omega}^m \bar{e}_n)}_{= \delta^m_n} (V^n \tilde{\omega}^j)$

$\Rightarrow B(\bar{e}_j,\bar{V})\tilde{\omega}^j = B_{jn} V^n \tilde{\omega}^j$

$= B_{jn} \tilde{\omega}^j V^n$

$= B(\ ,\bar{V})$

$\therefore C(\tilde{\sigma};\bar{V}) = A(\tilde{\sigma}, B(\ ,\bar{V}))$ → completely independent from basis.

Empty space

Aside: $B(\bar{e}_j, \bar{V})\tilde{\omega}^j = (B_{\ell m}\tilde{\omega}^\ell \tilde{\omega}^m)\ \bar{e}_j\ V^p\bar{e}_p\tilde{\omega}^j$

$$= B_{\ell m}\underbrace{(\tilde{\omega}^\ell\ \bar{e}_j)}_{\delta^\ell_j}\underbrace{(\tilde{\omega}^m\bar{e}_p)}_{\delta^m_p}V^p\tilde{\omega}^j$$

$$= B_{jp}V^p\tilde{\omega}^j$$

Recall: $\boxed{\tilde{\omega}^i\bar{e}_j = \delta^i_j}$

This is independent of basis and indices

§ 2.26 Basis Transformations

Recall, A tensor of type $\binom{N}{N'}$ is a linear function that takes N 1-forms and N' Vectors as arguments

This definition is modern. Previously tensors were defined as how components change under a change of Basis

At $P \in M$ suppose $\{\bar{e}_i\}$, $\{\bar{e}_{j'}\}$ are bases to T_pM. There is a linear transformation matrix Λ with $\bar{e}_{j'} = \Lambda^i_{j'}\bar{e}_i$

$\Lambda^i_{j'}$ is non-singular

Recall: One forms have a basis defined by $\tilde{\omega}^i(\bar{e}_k) = \delta^i_k$

We will determine how $\tilde{\omega}^i$ basis transforms. To do this, we multiply the definition by $\Lambda^k_{j'}$

$$\Lambda^k_{j'}\ \tilde{\omega}^i(\bar{e}_k) = \Lambda^k_{j'}\ \delta^i_k$$

$$\tilde{\omega}^i(\underbrace{\Lambda^k_{j'}\ \bar{e}_k}_{\text{by above}}) = \Lambda^i_{j'}$$

$$\tilde{\omega}^i(\bar{e}_{j'})$$

$$\Rightarrow \tilde{\omega}^i(\bar{e}_{j'}) = \Lambda^i_{j'}$$

We define the inverse of $\Lambda^i_{j'}$ ~~to~~ be $\Lambda^{k'}_j$

$$\boxed{\begin{aligned} \Lambda^{k'}_j\ \Lambda^j_{i'} &= \delta^{k'}_{i'} \\ \Lambda^{k'}_j\ \Lambda^i_{k'} &= \delta^i_j \end{aligned}}$$

We multiple the previous eqn by $\Lambda^{k'}_i$ ~~to~~ get $\Lambda^{k'}_i\ \tilde{\omega}(\bar{e}_{j'}) = \Lambda^{k'}_i\Lambda^i_{j'} = \delta^{k'}_{j'}$

$$\Lambda^{k'}_i\ \tilde{\omega}^i(\bar{e}_{j'}) = \delta^{k'}_{j'} = \tilde{\omega}^{k'}(\bar{e}_{j'})$$

The functions in front of $\bar{e}_{j'}$ must be equal.

$$\boxed{\tilde{\omega}^{k'} = \Lambda^{k'}_i\ \omega^i}$$

$\tilde{\omega}^{k'}$ transforms with $\Lambda^{k'}_i$

Compare with

$$\bar{e}_{j'} = \Lambda^{i}{}_{j'}\,\bar{e}_i \qquad \bar{e}_{j'} \text{ transfoms with } \Lambda^{i}{}_{j'}$$

1- forms transform with the inverse of Λ. we can also transform coordinates

$$V^{i'} = \tilde{\omega}^{i'}(\bar{V}) = \Lambda^{i'}{}_{j}\,\tilde{\omega}^{j}(\bar{V}) = \Lambda^{i'}{}_{j} V^{j}$$

and Similarly

$$q_{k'} = \tilde{q}(\bar{e}_{k'}) = \tilde{q}(\Lambda^{j}{}_{k'}\,\bar{e}_j) = \Lambda^{j}{}_{k'}\,\tilde{q}(\bar{e}_j) = \Lambda^{j}{}_{k'}\,q_j$$

Summary:

V^i and $\tilde{\omega}^i$ Transform with $\Lambda^{i'}{}_{j}$

q_i and $\bar{e}_i$ transform with $\Lambda^{i}{}_{k'}$

Its because of these difference that

$V^i \bar{e}_i$ & $V^j \bar{e}_j$ are basis independent.

Vector Coordinates (Superscipts) are contra variant since they transforms "oposite" to How There basis transforms

1- form Coordinates (subscripts) are covariant Since they transform Like $\bar{e}_i$

§ Coordinate Transformations

Suppose $U \subset M$ has coordinates $\{x^i, i=1,\dots,n\}$. Introduce new coordinates $\{y^i, i=1,\dots,n\}$

$$y^{j'} = f(x^1,\dots,x^n) \quad j' = 1,\dots,n$$

$$y^{j'} = f^{j'}(x^j)$$

Jacobian?

The coordinate transformation is $\dfrac{\partial y^{j'}}{\partial x^i}$ has a non-zero determinant in U

ALL $P \in V$ can be described with $\{x^i\}$ or $\{y^{j'}\}$ has coordinat vector basis

$$\left\{\frac{\partial}{\partial x^i}\right\} \text{ and } \left\{\frac{\partial}{\partial y^{j'}}\right\}$$

These must be related by

$$\frac{\partial}{\partial y^{j'}} = \frac{\partial x^i}{\partial y^{j'}} \frac{\partial}{\partial x^i}$$

Compare with what we saw previously

$$\bar{e}_{j'} = \Lambda^i{}_{j'} \, \bar{e}_i$$ explict expression for lamba

$$\therefore \quad \boxed{\Lambda^i{}_{j'} = \frac{\partial x^i}{\partial y^{j'}}}$$ covariant

Similarly $$\boxed{\Lambda^{k'}{}_j = \frac{\partial y^{k'}}{\partial x^j}}$$ Contra variant

Since $$\frac{\partial x^i}{\partial y^{j'}} \frac{\partial y^{j'}}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta^i_k$$

Lec 9 Oct 5, 2023

§ 2.27 Tensor Operations on Components

Given a tensor T with components $\{T^{i\dots}{}_{j\dots}\}$ on a basis, the following is basis invariant.

$$aT \quad \text{or} \quad \{a T^{i\dots}{}_{j\dots}\}$$

This can be denoted as $T \to aT$

The Outer Product is also basis invariant

$$A, B \to A \otimes B$$

OR

$$\{A^{i\dots}{}_{j\dots}\}, \{B^{k\dots}{}_{\ell\dots}\} \to \{A^{i\dots}{}_{j\dots} B^{k\dots}{}_{\ell\dots}\}$$

A tensor operation is one where operations on components produces components that are the tensor, independent of the basis. This include:

1. Addition
2. Scalar Mutiplication
3. Outer Products
4. Contractions

§ 2.28 Functions and Scalars

A scalar is a $\binom{0}{0}$ tensor, which is a function on M independent of the basis

Example: $V^i \tilde{\omega}_i$ is a scalar

V^i is not a scalar

§ 2.29 The metric tensor on a Vector Space

An inner product is a rule that associates a number with 2 vectors and it is a $\binom{0}{2}$ tensor

It is also reffered to as a metric tensor, $g_{\bar{1}}$ bar not one

$$\boxed{g_1(\bar{v}, \bar{w}) = g_1(\bar{w}, \bar{v}) \equiv \bar{w} \cdot \bar{v}}$$

g_1 is a symmetric tensor with components

$$\boxed{g_{ij} = g_1(\bar{e}_i, \bar{e}_j)}$$

We will require that g_1 has an inverse.

If $g_{ij} = \delta_{ij}$ then it is the Euclidean metric, and the vector space is Euclidean space., $\mathcal{E}^n$

Given any g_{ij}, we can change to a new basis, say $\{\bar{e}_{j'}\}$ such that

$$g_{i'j'} = \Lambda^k{}_{i'} \Lambda^l{}_{j'} \, g_{kl}$$

We can pick a basis where the metric tensor is diagonal and only has +1 or -1 as enteries

The convention is to list the -1's first then the +1's:

$$\boxed{g_{i'j'} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots 1)}$$

This is written in an orthanormal basis

The sum [trace] of these elements is the signature.

We classify the metric tensor as

positive definite if all +1's

negative definite if all -1's

indefinte if we have both 1's and -1's

Example: Minkowski metric in special relativity

$(-1, 1, 1, 1)$ or $(1, 1, 1, -1)$

This is indefinite

Euclidean Space, $\mathcal{E}^n$ is called Cartesian, with

$$g_{ij} = \delta_{ij} \quad \text{or} \quad g_1 = I$$

The minkowski metric has the Lorentz basis

$$\eta = \mathrm{diag}(-1, 1, 1, 1)$$

A transformation matrix from one Lorentz basis to another can be written as

$$\eta = \Lambda_L^T \, \eta \, \Lambda_L$$

The metric tensors maps vectors to 1- forms.

$$\tilde{V} = g_1(\bar{V}, \underbrace{\_}_{\text{Blank}})$$

In terms of Components,

$$
\begin{aligned}
V_i &= \tilde{V}(\bar{e}_i) \\
&= g_1(\bar{V}, \bar{e}_i) \\
&= g_1(V^j \bar{e}_j, \bar{e}_i) \\
&= V^j g_1(\bar{e}_j, \bar{e}_i) \\
&= \left.\begin{aligned} &V^j g_{ji} \\ &g_{ij} V^j \end{aligned}\right\} \text{Symmetry in } g_{ij}
\end{aligned}
$$

The inverse matrix g_{ij} is called g^{ij} with

$$g^{ij} g_{jk} = \delta^i_k \quad \text{or} \quad \overset{\text{you can show}}{g_{ij} \, g^{jk} = \delta^k_i}$$

This yields,

$$
\begin{aligned}
&g^{ki} V_i \\
&= g^{ki} g_{ij} V^j \\
&= \delta^k_j V^j
\end{aligned}
$$

This recovers what we had before

Summary: $V_i = g_{ij} V^j$

and

$V^k = g^{ki} V_i$

A $\binom{2}{0}$ tensor, A, can map to a $\binom{1}{1}$ tensor

$$A^i{}_j = g_{jk} A^{ik}$$

This can be mapped to $\binom{0}{2}$ tensor

$$A_{\ell j} = g_{\ell m} A^m{}_j$$

This can be inverted $A^{ik} = g^{i\ell} g^{km} A_{\ell m}$

Skip the proof

This is called index raising and lowering with a metric tensor, there is much less diff difference betwen $\binom{N}{N'}$ and $\binom{N-1}{N'-1}$ and $\binom{N+1}{N'+1}$

Hence why we often refer to them as tensors of order $N+N'$

In a Euclidean vector space, a cartesian basis is $g_{ij} = g^{ij}$

There is no difference between superscripts and subscripts. and hence we often only use subscripts.

§ 2.30 The metric tensor field on a manifold

A metric tensor g, on a manifold is a $\binom{0}{2}$ symmetric tensor and has an inverse at every point.

For all points in the manifold, g serves as a metric on T_pM and has all the propreties mentioned previously. but there's more

Using g, we can define distance and curvature we can use g_1 to define length on M

Suppose a curve has a tangent vector $\bar{V} = \frac{d\bar{x}}{d\lambda}$

then
$$dl^2 = d\bar{x} \cdot d\bar{x} = (\bar{V} d\lambda)\cdot(\bar{V} d\lambda)$$
$$= \bar{V}\cdot\bar{V}\,(d\lambda)^2$$
$$= g_1(\bar{V},\bar{V})\,d\lambda^2$$

d is infinitesmal and not gradient

If g_1 is positive definite then dl^2 is positive and

$$\boxed{dl = (g_1(\bar{V},\bar{V}))^{\frac{1}{2}}\,d\lambda}$$

If g_1 is in definite then curves can have dl positive (space like) or negative (time like)

dl is the proper distance for space like curves and the proper time for time-like curves

§ 2.31 Special Relativity

$\mathbb{R}^4$ with a metric with signature +2 is a manifold called Minkowski space time from special relativity

We can define coordinates $(\Delta t, \Delta x, \Delta y, \Delta z)$ then
$$\Delta s^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

c is the speed of light

Define $x^0 = ct$, $x^1 = x$ $x^2 = y$, $x^3 = z$ then

$$\Delta s^2 = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2$$

or $$\Delta s^2 = \eta_{\alpha\beta}\,\Delta x^\alpha \Delta x^\beta$$

This is a psuedo Norm and satisfies

(N$_{ii}$) factor out a Scalar
(Niv) parallelgram rule

These are what we need to define an inner product

$$\bar{V} \cdot \bar{W} = \eta_{\alpha\beta} V^{\alpha} W^{\beta}$$

Midterm: covers lec 1-9, but none of the special Relativity

To study: make sure you understand all the lectures and assigments [soln's on Sunday]

Francis will give a description of questions to expect

Will post sample formula Sheet on Monday.

Lec 11 ~ Oct 19

§ 3.1 Intro: How a vector field maps a manifold to itself

Recall: A vector field $\bar{V}$ induces an integral curve

$$\boxed{\frac{dx^i}{d\lambda} = v^i(x^j)}$$ → all possible cordinates

Propreties of integral Curves:

① $\exists$ a unique curve through each $P \in M$

② These curves fill the manifold M

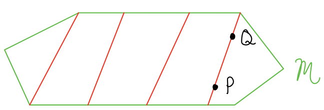
If M is n dimensonal then the set of integral curves are (n-1) dimensional

to cover the entire manifold curve has 1 parameter, need (n-1) more to get n-manifold

Curves like this that fill the manifold is a congruence.

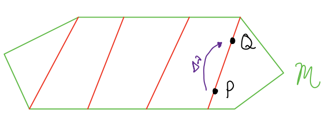
These integral Curves provide a natural mapping from M to M along $\bar{V}$. If $\bar{V}$ is C^∞, then the mapping is diffeomorphic

Such a mapping is Lie dragging



§ 3.2 Lie dragging a function.

Suppose f is a function on a manifold M.



P and Q are on the same Curve

We define $\boxed{f(P) = f^*_{\Delta\lambda}(Q)}$ → How to get to P from Q along an integral Curve

Star denotes new

$\Delta\lambda$ is very small, but could be big, If $f(Q) = f^*_{\Delta\lambda}(P)$ then f is invariant under the map.

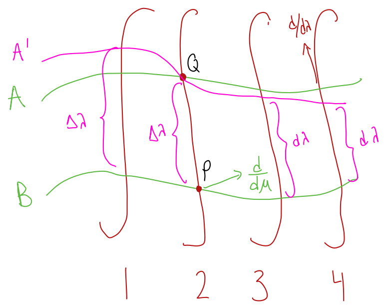
If f is invariant $\forall \Delta\lambda$, then it is said to be Lie dragged

If function f that is Lie dragged must satisfy, $\frac{df}{d\lambda} = 0$

§ 3.3 Lie dragging a Vector field

A vector field can be defined by a congruence of curves for which it is tangent.

We now show how to Lie drag a vector field



1, 2, 3, 4 are Integral curves of $\frac{d}{d\lambda}$ (Congruence in λ)

A, B are Integral curves of $\frac{d}{d\mu}$ They form a congruence of μ

The points along A (μ congruence) are dragged along $\Delta\lambda$ to the Curve A', A' need not be a congruence of μ

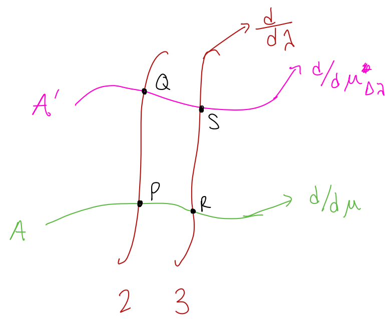
A' defines a new congruence with parameter $\mu^*_{\Delta\lambda}$

This has a tangent Vector field $\frac{d}{d\mu^*_{\Delta\lambda}}$, which is the image of $\frac{d}{d\mu}$ under Lie dragging

In general $\mu^*_{\Delta\lambda}$ congruence differs from μ congruence. If they are the same then $\boxed{\frac{d}{d\mu^*_{\Delta\lambda}} = \frac{d}{d\mu}}$ every where

We say a vector field and congruence are invariant under the map

If it is invariant $\forall \Delta\lambda$, then the curves are said to be Lie dragged by $d/d\lambda$.



If the distances are infinitesemal and if $\frac{d}{d\mu}$ stretches from P to R on the curve A then:

$\frac{d}{d\mu^*_{\Delta\lambda}}$ Streches from Q to S on A'

If $\frac{d}{d\mu}$ is lie draged then B conincides with A and

$$\left(\frac{d}{d\mu^*}{}_{\Delta\lambda}\right)_Q = \left(\frac{d}{d\mu}\right)_Q$$

This implies $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = 0$

A vector field is Lie dragged iff $\left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right] = 0$

§ 3.4 Lie derivative.

The derivative of a scalar valued function $\mathbb{R}$ is:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \Big\} \text{ newton's quotient}$$

To compute this we need to compare the function at different points and divide by the distance between them. This has 2 problems:

① We don't always have distance but we have the parameter λ along 2 Integral curves

② Must compare function at different points
We do this be lie dragging

① function:

Method ① evaluate f at $\lambda_0 + \Delta\lambda$, $f(\lambda_0 + \Delta\lambda)$ and drag it back to λ_0

② Evaluate f at λ_0

③ Find the difference, $\div$ by $\Delta\lambda$ and take the limit as $\Delta\lambda \to 0$

For ① define f^* such that $\frac{df^*}{d\lambda} = 0$, hence $f^*(\lambda_0) = f(\lambda_0 + \Delta\lambda)$

Hence we get:

$$\lim_{\Delta\lambda \to 0} \frac{f^*(\lambda_0) - f(\lambda_0)}{\Delta\lambda}$$

$$= \lim_{\Delta\lambda \to 0} \frac{f(\lambda_0 + \Delta\lambda) - f(\lambda_0)}{\Delta\lambda} = \left.\frac{df}{d\lambda}\right|_{\lambda_0}$$

The lie derivative of a function is $\boxed{\mathcal{L}_{\bar{V}} f = \bar{V}(f) = \frac{df}{d\lambda}}$

Lec 12 - Oct 24

§ 3.4 Lie derivates (Continued)

The lie derivative of a function $f: M \to \mathbb{R}$ along a vector field $\bar{V}$ is computed using Lie dragging

Recall, $f(P) = f^*_{\Delta\lambda}(Q)$ and $f(\lambda_0) = f^*_{\Delta\lambda}(\lambda_0 + \Delta\lambda)$

If we $\Delta\lambda \longrightarrow -\Delta\lambda$ and $\lambda_0 \longrightarrow \lambda_0 + \Delta\lambda$ then

$$f(\lambda_0) = f^*_{-\Delta\lambda}(\lambda_0 - \Delta\lambda)$$

$$f(\lambda_0 + \Delta\lambda) = f^*_{-\Delta\lambda}(\lambda_0)$$

Then the Lie derivative of f along $\bar{V}$ is

$$(\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \lim_{\Delta\lambda \to 0} \frac{f^*_{-\Delta\lambda}(\lambda_0) - f(\lambda_0)}{\Delta\lambda}$$

$$= \lim_{\Delta\lambda \to 0} \frac{f(\lambda_0 + \Delta\lambda) - f(\lambda_0)}{\Delta\lambda}$$

$$\Rightarrow (\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \left[\frac{df}{d\lambda}\right]_{\lambda_0} = \bar{V}(f)_{\lambda_0}$$

In component form $(\mathcal{L}_{\bar{V}} f)_{\lambda_0} = \left(\frac{dx^i}{d\lambda}\frac{\partial f}{\partial x^i}\right)_{\lambda_0}$

This generalizes to any differentiable manifold

Most text book's use push back / pull forwards.

what is a derivative along a vector?

To Compute the Lie derivative of a vector field, consider $\bar{U} = \frac{d}{d\mu}$ and $\bar{V} = \frac{d}{d\lambda}$ and consider an arbitrary function f

At λ_0 and $\lambda_0 + \Delta\lambda$ we know that the Lie derivative is

$$(\mathcal{L}_{\bar{U}} f)_{\lambda_0} = \left(\frac{df}{d\mu}\right)_{\lambda_0} = (\bar{U}(f))_{\lambda_0}$$

$$(\mathcal{L}_{\bar{U}} f)_{\lambda_0 + \Delta\lambda} = \left(\frac{df}{d\mu}\right)_{\lambda_0 + \Delta\lambda} = (\bar{U}(f))_{\lambda_0 + \Delta\lambda}$$

Now, we can Lie drag $\bar{U}(\lambda_0 + \Delta\lambda)$ to λ_0 with $\bar{U}^*_{-\Delta\lambda}(\lambda_0) = \bar{U}(\lambda_0 + \Delta\lambda) = \frac{d}{d\mu^*_{-\Delta\lambda}}$

With $[\bar{U}^*_{-\Delta\lambda}, \bar{V}] = 0$ because of what francis said in lecture [transcribe later]

We taylor expand $\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0}$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0 + \Delta\lambda} = \left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} + \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0} + O(\Delta\lambda^2)$$

If we Solve for the first term on the RHS,

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0 + \Delta\lambda} - \Delta\lambda\left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0}$$

BUT $\frac{d}{d\mu^*_{-\Delta\lambda}} = \frac{d}{d\mu}$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{df}{d\mu}\right]_{\lambda_0+\Delta\lambda} - \Delta\lambda \left[\frac{d}{d\lambda}\left(\frac{d}{d\mu^*_{-\Delta\lambda}} f\right)\right]_{\lambda_0}$$

↳ Taylor Expand this term

Change order because $[\bar{u}^*_{-\Delta\lambda}, \bar{v}] = 0$

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} = \left[\frac{df}{d\mu}\right]_{\lambda_0} + \Delta\lambda \left[\frac{d}{d\lambda}\left(\frac{df}{d\mu}\right)\right]_{\lambda_0} - \Delta\lambda \left[\frac{d}{d\mu^*_{-\Delta\lambda}}\left(\frac{df}{d\lambda}\right)\right]_{\lambda_0} + O(\Delta\lambda^2)$$

We define the Lie derivative of $\bar{u}$ along $\bar{v}$ is

$$(\mathcal{L}_{\bar{v}} \bar{u})(f) = \lim_{\Delta\lambda\to 0} \left[\frac{\bar{u}_{-\Delta\lambda}(\lambda_0) - \bar{u}(\lambda_0)}{\Delta\lambda}\right](f)$$

$$= \lim_{\Delta\lambda\to 0} \left[\left(\frac{df}{d\mu^*_{-\Delta\lambda}}\right)_{\lambda_0} - \left(\frac{df}{d\mu}\right)_{\lambda_0}\right] / \Delta\lambda$$

Aside ✻

$$\left[\frac{d}{d\mu^*_{-\Delta\lambda}} f\right]_{\lambda_0} - \left[\frac{df}{d\mu}\right]_{\lambda_0} = \Delta\lambda \left[\frac{d}{d\lambda}\left(\frac{df}{d\mu}\right)\right]_{\lambda_0} - \Delta\lambda \left[\frac{d}{d\mu^*_{-\Delta\lambda}}\left(\frac{df}{d\lambda}\right)\right]_{\lambda_0} + O(\Delta\lambda^2)$$

Substitute ✻

$$(\mathcal{L}_{\bar{v}} \bar{u})(f) = \lim_{\Delta\lambda\to 0} \left[\frac{d}{d\lambda}\frac{d}{d\mu} f - \frac{d}{d\mu^*_{-\Delta\lambda}}\frac{d}{d\lambda} f\right] + O(\Delta\lambda)$$

This is $O(\Delta\lambda)$ b/c $\bar{\omega}$ divided by $\Delta\lambda$ in the limit

$$(\mathcal{L}_{\bar{v}} \bar{u})(f) = \left[\frac{d}{d\lambda}, \frac{d}{d\mu}\right](f) \quad \text{or} \quad = [\bar{v}, \bar{u}](f)$$

Lie bracket is antisymmetric

This is equivalent to the directional derivative of $\bar{u}$ in the direction of $\bar{v}$

§ 3.5 Lie derivative of a one-form

We can determine the Lie derivative of a 1-form in terms of the Lie derivative of a function and Vector field

The Lie derivative of a One form can be computed as follows:

Libentz rule is Product Rule

$$\mathcal{L}_{\bar{v}}[\tilde{\omega}(\bar{w})] = \frac{d}{d\lambda}[\tilde{\omega}(\bar{w})]$$

$$= \frac{d\tilde{\omega}}{d\lambda}(\bar{w}) + \tilde{\omega}\left(\frac{d\bar{W}}{d\lambda}\right)$$

OR

$$\mathcal{L}_{\bar{v}}[\tilde{\omega}(\bar{w})] = (\mathcal{L}_{\bar{v}}\tilde{\omega})\bar{w} + \tilde{\omega}(\mathcal{L}_{\bar{v}}\bar{W})$$

This is the Operator, there is an imaginary function at the end of all these terms

This method extends to the outer product of tensors

$$\mathcal{L}_{\bar{v}}(A \otimes B) = \mathcal{L}_{\bar{v}}(A) \otimes B + A \otimes \mathcal{L}_{\bar{v}}(B)$$

Or,

$$\mathcal{L}_{\bar{v}}(T(\tilde{\omega},\dots;\bar{v},\dots)) = (\mathcal{L}_{\bar{v}} T)(\tilde{\omega},\dots;\bar{v},\dots) + T(\mathcal{L}_{\bar{v}}\tilde{\omega},\dots;\bar{v},\dots) + \dots + T(\tilde{\omega},\dots;\mathcal{L}_{\bar{v}}(\bar{v}),\dots) + \dots$$

for all components of the tensor

This is the Product or Leibniz Rule

§ 3.6 Submanifold

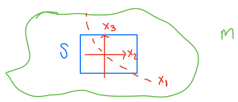
The idea is that a submanifold S of a manifold M is a subset of M which is itself a manifold.

Ex $\mathbb{R}^3$ is a manifold, a smooth surface is a submanifold.
a smooth curve is a submanifold

A m-dimensional submanifold S of a n-dimensional is a subset of M with the proprety that in some neighbourhood of P

$P \in S \subset M$, there exists a coordinate system for M in which the points of S can be written as

$$x^1 = x^2 = \dots = x^{n-m} = 0$$



Solns to the system of DEs $y_i = f_i(x^1, \dots, x^n)$ with $i = 1, \dots, p$ and with coordinates $\{x^1, \dots, x^n\}$. This is a submanifold with coordinates $\{y_1, \dots, y_n, x^1, \dots, x^n\}$

Suppose $P \in S \subset M$ with $\dim S = m$ and $\dim M = n$. A curve in S through P is a curve in both M & S, through P.

T_pS: Tangent space at P in S (dim m)

T_pM: Tangent space at P in M (dim n)

T_pS is a vector subspace of T_pM and a submanifold

A tangent vector at P is both in T_pS and T_pM

T_p^*S : cotangent space at P in S
T_p^*M: cotangent space at P in M

Any $\tilde{\omega} \in T_p^*M$ yields a $\tilde{\omega} \in T_p^*S$ if we restrict the domain to T_pS instead of T_pM.

However, $\tilde{\omega} \in T_p^*S$ does not yield a unique $\tilde{\omega}$ in T_p^*M

Summary: $\bar{V} \in T_pS$ is also a vector in T_pM and $\tilde{\omega} \in T_p^*M$ is also a one form in T_p^*S

Lec 13 - Oct 26th

Lie derivatives

$(\mathcal{L}_{\bar{V}} \bar{U})^i \quad V^r \frac{\partial}{\partial x^r} U^i - U^r \frac{\partial}{\partial x^r} V^i$

$(\mathcal{L}_{\bar{V}} \tilde{\omega})_i = V^r \frac{\partial}{\partial x^r} \omega_i + \omega_r \frac{\partial}{\partial x^i} V^r$

indices in between

In general for $T^{i \dots j}{}_{k \dots l}$

$$(\mathcal{L}_{\bar{V}} T)^{i \dots j}{}_{k \dots l} = V^r \frac{\partial}{\partial x^r} T^{i \dots j}{}_{k \dots l} - T^{r \dots j}{}_{k \dots l} \frac{\partial}{\partial x^r} V^i - \dots - T^{i \dots r}{}_{k \dots l} \frac{\partial}{\partial x^r} V^j + T^{i \dots j}{}_{r \dots l} \frac{\partial}{\partial x^k} V^r + \dots + T^{i \dots j}{}_{k \dots r} \frac{\partial}{\partial x^l} V^r$$

§ 3.7 Frobenius Thm (Vector Field Version)

Suppose a coordinate patch of $S \subset M$ has coordinates y^a $a = \{1, \dots, n\}$ with basis vectors $\{\frac{\partial}{\partial y^a}\}$ for vector fields on S, with $\left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b}\right] = 0$ $\forall a \neq b$. Since it is a coordinate basis

It can be shown that in general for a non coordinate basis Λ, the Lie bracket of any of these two vector fields, yields a vector field tangent to S

The next theorem says something about the sub manifold if we know a proprety of the Lie bracket of a Vector Field.

Frobenius' theorem (Vector Field Version)

If a set of m smooth vector feilds in $U \subset M$ (subset) have Lie brackets which is a Linear Combinations of the m vector field, then the integral curves of the fields mesh to form a family of sub manifolds

means combined

Implications

Dim of the submanifold is $\leq m$

Each point in U is on one and only on submanifold. This family of sub manifold is a foiliation of U and fills U like the congruence curve do. Each sub manifold is a leaf.

§ 3.9 An Example: the generation of S^2

Consider a ϕ-based vector in spherical coordinates called $\bar{e}_\phi = -y\,\bar{e}_x + x\,\bar{e}_y$

Using our notion, this becomes

$$\frac{\partial}{\partial \phi} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \vec{l}_z$$

angular momentum operator in the z direction

Similarily
$$\vec{l}_x = -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}$$
$$\bar{l}_y = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$$

It can be shown
$$[\bar{l}_x, \bar{l}_y] = -\bar{l}_z$$
$$[\bar{l}_y, \bar{l}_z] = -\bar{l}_x$$
$$[\vec{l}_3, \bar{l}_x] = -\bar{l}_y$$

Checking $[\bar{l}_x, \bar{l}_y] = \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right) - \left(-x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}\right)\left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)$

$$\vdots$$
$$= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} = -\bar{l}_z$$

Since $\{\bar{l}_x, \bar{l}_y, \bar{l}_z\}$ have Lie brackets that are a linear combination of the set, Frobenius' theorem yields integral curves that form a Submanifold

Since we have 3 vector fields, we might think the dimension of this set is 3, it turns out the dim is 2.

To see this consider $r = (x^2+y^2+z^2)^{1/2}$

We can show that $\tilde{d}r$ is the gradient of r
$$\tilde{d}r(\bar{l}_x) = \tilde{d}r(\bar{l}_y) = \tilde{d}r(\bar{l}_z) = 0$$

check: $\tilde{d}r(\bar{l}_x) = \bar{l}_x(r)$
$$= \left(-z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$
$$= -z\left(\frac{y}{r}\right) + y\frac{z}{r} = 0$$

B/c of symmetry of the operators, these all equal zero.

Since the gradients are 0 these exist in the tangent space $\Rightarrow$ dim 2

We can consider $\tilde{d}r$ to be a set of surfaces of constant r, S^2. Since $\bar{l}_x, \bar{l}_y, \bar{l}_z$ are orthogonal to the gradient, they must all lie in the tangent, which is 2 dim $\therefore$ $\{\bar{l}_x, \bar{l}_y, \bar{l}_z\}$ is only dimenson 2

§ 3.10 Invariance

Lie derivatives are often used to show that a tensor is invariant in a direction

We say T is invariant under a vector field $\bar{V}$ if
$$\mathcal{L}_{\bar{V}} T = 0$$

T could be
1. metric tensor
2. a Scalar field for P E of a partical
3. Vectors $\bar{V}$ under which T is invariant are important.

§ 3.11 killing Vector fields

Metric tensor can be invariant with respect to a vector field. These Vector fields are important.

A killing Vector field is a vector field, $\bar{V}$, such that

$$\mathcal{L}_{\bar{v}}\, g_1 = 0$$

From excersize 3.4, you deduce that $(\mathcal{L}_{\bar{v}}\, g_1)_{ij} = V^k \frac{\partial}{\partial x^k} g_{ij} + g_{kj} \frac{\partial}{\partial x^i} V^k + g_{ik} \frac{\partial}{\partial x^j} V^k = 0$

For a killing vector field

Pick coordinates such that the integral curve are in the x^1 direction, then

$$V^i = \delta^i_1$$

Then the above simplifies,

$$(\mathcal{L}_{\bar{v}}\, g_1)_{ij} = \frac{\partial}{\partial x^1} g_{ij} = 0$$

Therefore the metric tensor is invariant with respect to the killing vector

Example: Consider $\mathbb{R}^3$ in the different coordinates

1. Euclidean space $\quad g_{ij} = \delta_{ij}$

This form is independent of x, y, z and $\therefore$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are all killing vectors

2. Spherical coordinats $\quad g_{rr} = \frac{\partial}{\partial r} \cdot \frac{\partial}{\partial r} = 1$

$$g_{\theta\theta} = \frac{\partial}{\partial \theta} \cdot \frac{\partial}{\partial \theta} = r$$

$$g_{\phi\phi} = \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} = r^2 \sin^2\theta$$

→ g_1 is a $\binom{0}{2}$ tensor and only diagond is non zero

g_1 is independent of ϕ hence $\bar{l}_z$ is a killing vector

It can be shown that $\bar{l}_x$ and $\bar{l}_y$ are also killing vectors

These 6 killing vectors $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \bar{l}_x, \bar{l}_y, \bar{l}_z$ are the only killing vectors possible

§ 3.12 killing Vectors and Conserved quantities in partical dynamics.

In classical mechanics it follow that:

1. if the PE function is axially symmetric, then the angular momentum is consered

② If the PE is independent of say x, the the x component of moment is Conserved

these symmetries in the PE energy function give rise to conserved quantities.

However if another Symmetry is found in PE function does that mean some thing else is Conserved?

Conserved quantities dont just require the PE is invariant w.r.t to a variable but we also require that is a killing vector

Idea: Newton's 2nd Law

$$m\dot{\vec{V}} = -\vec{\nabla}\Phi \quad \text{or} \quad m\dot{V}^i = -\vec{\nabla}^i\Phi \quad \text{or} \quad m\dot{V}^i = -g^{ij}\frac{\partial}{\partial x^j}\Phi$$

Any invariance of this equation that both Φ and g_1^{-1} are invariant w.r.t coordinate

Lec 14 - Oct 13th

Claim: most abstract concepts have been introduced. Chapter 4 is on differential forms

§4 Differential forms

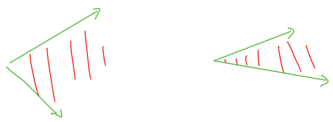
Now, we develop calculus of differential forms or often called exterior calculus or differentiable manifolds

A The algebraic & integral calculus of forms

§ 4.1 Defn of Volume and the geometric role of differential forms

we now study a class of tensors that enable us to define volume on elements (without an inner product)

A pair of non parallel vectors in euclidean space defines an infinitesmial area

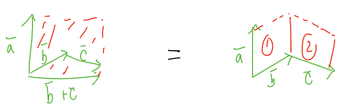


In our definition of area (volume), we do not need to know the length of the vector or the angle between th

Consider a 2D manifold and suppose we have two linearly independent infinitesmial vectors, They form a parralelogram

We want to find the area between $\bar{a}, \bar{b}$. Our definition of area must satisfy the following $\bar{a}$ $\bar{b}$ $\bar{c}$

$\boxed{\text{area}(\bar{a},\bar{b}) + \text{area}(\bar{a},\bar{c}) = \text{area}(\bar{a}, \bar{b}+\bar{c})}$



Since area(,) takes 2 vectors and yields a number it must be a $\binom{0}{2}$ tensor

Observe, $\boxed{\text{area}(\bar{u},\bar{a}) = 0 \text{ for all } \bar{a}}$

Excersize 4.1 If B is a $\binom{0}{2}$ tensor with $B(\bar{u},\bar{u}) = 0$ $\forall$ $\bar{U}$ then $B(\bar{u},\bar{w}) = -B(\bar{w},\bar{u})$

Proof $B(\bar{u}+\bar{w}, \bar{u}+\bar{w}) = 0$

B/c Linear operator

$B(\cancel{\bar{u},\bar{u}})^{0} + B(\bar{w},\bar{u}) + B(\bar{u},\bar{w}) + B(\cancel{\bar{w},\bar{w}})^{0} = 0$

$B(\bar{u},\bar{w}) = -B(\bar{w},\bar{u})$

Note ① that the area function must satisfy this anti symmetry proprety
② Area is not non negative

Recall for Linear algebra, we can find area using

$$\text{aree} = \det \begin{vmatrix} V_x & V_y \\ W_x & W_y \end{vmatrix}$$ Anti symetric

§ 4.2 Notation and definitions for anti symmetric tensors

A $\binom{0}{2}$ tensor is anti symetric if

$$\tilde{\omega}(\bar{u}, \bar{v}) = -\tilde{\omega}(\bar{v}, \bar{u})$$

A $\binom{0}{3}$ tensor is anti symetric if it changes signs when we exchange any 2 elements

$$\begin{aligned} \tilde{\omega}(\bar{u}, \bar{v}, \bar{w}) &= -\tilde{\omega}(\bar{v}, \bar{u}, \bar{w}) \\ &= -\tilde{\omega}(\bar{w}, \bar{v}, \bar{u}) \\ &= -\tilde{\omega}(\bar{v}, \bar{w}, \bar{v}) \end{aligned}$$

Given any tensor we can build an antisymmetric version of it.

Ex. If $\tilde{\omega}$ a $\binom{0}{2}$ tensor then

$$\tilde{\omega}_A(\bar{u}, \bar{v}) = \frac{1}{2!}\left[\tilde{\omega}(\bar{u}, \bar{v}) - \tilde{\omega}(\bar{v}, \bar{u})\right]$$

This is the anti symmetric part of $\tilde{\omega}$

If $\tilde{p}$ is a $\binom{0}{3}$ tensor then

$$\tilde{p}_A(\bar{u}, \bar{v}, \bar{w}) = \frac{1}{3!}\left[\tilde{p}(\overset{1}{\bar{u}}, \overset{2}{\bar{v}}, \overset{3}{\bar{w}}) + \tilde{p}(\overset{2}{\bar{v}}, \overset{3}{\bar{w}}, \overset{1}{\bar{u}}) + \tilde{p}(\overset{3}{\bar{w}}, \overset{1}{\bar{u}}, \overset{2}{\bar{v}}) - \tilde{p}(\overset{3}{\bar{w}}, \overset{2}{\bar{v}}, \overset{1}{\bar{u}}) - \tilde{p}(\overset{2}{\bar{v}}, \overset{1}{\bar{u}}, \overset{3}{\bar{w}}) - \tilde{p}(\overset{1}{\bar{u}}, \overset{3}{\bar{w}}, \overset{2}{\bar{v}})\right]$$

Normalizing by number of terms

$\tilde{p}_A$ is the anti symmetric part of $\tilde{p}$

Notation:

$$(\tilde{\omega}_A)_{ij} = \frac{1}{2!}(\omega_{ij} - \omega_{ji}) \equiv \omega_{[ij]}$$ This denotes antisymmetric part of $\tilde{\omega}$

$$(\tilde{p}_A)_{ijk} = \frac{1}{3!}(P_{ijk} + P_{jki} + P_{kij} - P_{kji} - P_{jik} - P_{ikj}) \equiv P_{[ijk]}$$

$[i \ldots k]$ denotes a completely anti symmetric set of indicies

Notation: we use $\sim$ to denot a complety anti symetric part of a tensor.

example: T is only at tensor & $\tilde{T}$ is the anti symmetric version of T

Also we say a one form is anti symmetric

Property: For an n-dimensional vector space, a completely antisymmetric $\binom{0}{p}$ tensor $(p \leq n)$ has at most

How many ways can we pick p directions in n space? $C_p^n = \frac{n!}{p!(n-p)!} = \binom{n}{p}$ "n choose p" independent components

Why be anti symmetric? will learn later on!

ex In $\mathbb{R}^3$ $n=3$ $C_1^3 = 3$, $C_2^3 = 3$, $C_3^3 = 1$

x | xy | x,y,z
y | xz |
z | yz |

§ 4.3 Differential forms

A p-form $(p \geq 2)$ is a completely anti symmetric tensor of type $\binom{0}{p}$

A one-form is a $\binom{0}{1}$ tensor (by convention anti symmetric)

A zero-form is a $\binom{0}{0}$ tensor (scalar) ???

$\Rightarrow$ p is the degree

Using $\otimes$ (outer product) can take 2 $\binom{0}{1}$ forms to yield a $\binom{0}{2}$ tensor.

A wedge product takes two one forms and yields a 2-form.

i claim this is anti symmetric

$$\boxed{\tilde{p} \wedge \tilde{q} = \tilde{p} \otimes \tilde{q} - \tilde{q} \otimes \tilde{p}} = -\tilde{q} \wedge \tilde{p}$$

$$\tilde{q} \wedge \tilde{p} = \tilde{q} \otimes \tilde{p} - \tilde{p} \otimes \tilde{q}$$

Proprety: If $\{\vec{e}_i = i = 1, \dots, n\}$ is a basis of T_pM and $\{\tilde{\omega}^j\}$ is the dual basis of $T_p{*}M$

then $\{\tilde{\omega}^j \wedge \tilde{\omega}^k, j,k = 1 \dots n\}$ is a basis for the vector space of two-forms

We can build two-forms in a similar way

$$\tilde{p} \wedge (\tilde{q} \wedge \tilde{R}) = (\tilde{p} \wedge \tilde{q}) \wedge \tilde{R} = \tilde{p} \wedge \tilde{q} \wedge \tilde{r}$$

$$\tilde{p} \wedge \tilde{q} \wedge \tilde{R} = \tilde{p} \otimes \tilde{q} \otimes \tilde{r} + \tilde{q} \otimes \tilde{r} \otimes \tilde{p} + \tilde{r} \otimes \tilde{p} \otimes \tilde{q} - \tilde{r} \otimes \tilde{q} \otimes \tilde{p} - \tilde{q} \otimes \tilde{p} \otimes \tilde{r} - \tilde{p} \otimes \tilde{r} \otimes \tilde{q}$$

We can define the wedge product of a p-form and a q-form

§ 4.4 Manipulating differential forms

Commutation rule of form: $\tilde{p} \wedge \tilde{q} = \tilde{q} \wedge \tilde{p} \, (-1)^{pq}$

idea: if $\tilde{p} = \tilde{\omega}^i \wedge \dots \wedge \tilde{\omega}^j$ p factors

$\tilde{q} = \tilde{\omega}^k \wedge \dots \wedge \tilde{\omega}^l$ q factors

$$\tilde{p} \wedge \tilde{q} = (\tilde{\omega}^i \wedge \dots \wedge \omega^j) \wedge (\tilde{\omega}^k \wedge \dots \wedge \tilde{\omega}^l)$$

Lecture 15 - Nov 2, 2023

§ 4.4 Manipulation of differential forms

Commutation rule: $\tilde{p} \wedge \tilde{q} = (-1)^{pq}\, \tilde{q} \wedge \tilde{p}$

Interior Product / Contraction of a vector with a form

If $\tilde{\alpha}$ is a p-form and $\bar{v}$ is a vector, then $\tilde{\alpha}$ requires p vectors

$\tilde{\alpha}(\bar{v}) \equiv \tilde{\alpha}(\bar{v}, \underbrace{\ldots}_{p-1 \text{ arguments left}})$ or $\alpha_{ij\cdots k} V^i$ ↗ text books notation

$i_{\bar{v}}(\tilde{\alpha}) = \alpha_{ij\cdots k} V^i$ → other text books notation

This is an inner product

Example:

$$i_{\bar{v}}(\tilde{\omega}^i \wedge \tilde{\omega}^j \wedge \cdots \wedge \tilde{\omega}^k) = V^i \tilde{\omega}^j \wedge \cdots \wedge \tilde{\omega}^k - V^j \tilde{\omega}^i \wedge \cdots \wedge \tilde{\omega}^k + \ldots$$

→ more terms check text book

§ 4.5 Restrictions to forms

Suppose W is a subspace of a Vector field V. A p-form, $\tilde{\alpha}$, is a $\binom{0}{p}$ tensor that is (completely) antisymmetric and its arguments could be

$$\underbrace{V \times V \times \cdots \times V}_{p\text{-times}}$$

The restriction of $\tilde{\alpha}$ to the subspace $W \subset$ is the same p-form but with the domain restricted to W,

$$\boxed{\tilde{\alpha}\big|_W (\bar{X}, \ldots, \bar{Y}) = \tilde{\alpha}(\bar{X}, \ldots, \bar{Y}) \quad \text{where} \quad \bar{X}, \ldots, \bar{Y} \text{ are in } W}$$

If $m = \dim W < p$ then $\tilde{\alpha}|_W$ is 0

If $m = p$ then $\tilde{\alpha}|_W$ has one component, $C^p_p = 1$

Restricted a form is called sectioning

A form is anulled by a vector space if its restriction to it vanishes

§ 4.6 Fields of forms

A field of p-forms on manifold M gives a p-form $\forall$ points on the manifold M.

A Sub manifold $S \subset M$ picks a subspace $T_p S$ for all $p \in S$ and we define the restriction of the p-form $\tilde{\alpha}$ to S by restricting $\tilde{\alpha}$ at P to $T_P S$.

§ 4.7 Handedness and Orientiability

In a n-dimensional manifold there is a 1-dimensional space of n-forms $(C^n_n = 1)$

Suppose that $\tilde{\omega}$ is an n-form field that we can use to find the Volume If we have $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a Vector basis of $T_P M$ that is Linearly Independent

It follows that $\tilde{\omega}(\bar{e}_1, \ldots, \bar{e}_n) \neq 0$ iff $\tilde{\omega} \neq 0$ at P

Aside: $\tilde{\omega} \Rightarrow \omega_{i \ldots k} \tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge \ldots \wedge \tilde{\omega}_n$

Consider

$$\tilde{\omega}(\bar{e}_1, \ldots, \bar{e}_n) = \omega_{i \ldots k} (\tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \ldots \wedge \tilde{\omega}^n)(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$$

$$= \omega_{i \ldots k} \tilde{\omega}^1(\bar{e}_1) \wedge \tilde{\omega}^2(\bar{e}_2) \wedge \ldots \tilde{\omega}^n(\bar{e}_n)$$

$$= \omega_{i \ldots k}$$

$\tilde{\omega}$ seperates the vector bases in to 2 classes

① $\tilde{\omega}(\bar{e}_1, \ldots, \bar{e}_n) > 0$ (right hand)

② $\tilde{\omega}(\bar{e}_1, \ldots, \bar{e}_n) < 0$ (left hand)

This seperation is unique to any n-form.

This manifold is said to be orientable if we define the handedness consistently (all positive or all negative) on the manifold

Example $\mathbb{R}^n$ is orientable

Mobius is not orientable

We only consider orientable manifold

§ 4.8 Volumes and Integration on Oriented Manifolds

A set of n linearly independent vectors (infinetesimal) or an n-dim manifold, can define a non-zero volume. this forms in n-dim a parallelepiped

Integration a function f on M requires multiply f by an infinitesimal Volume then adding this up over all of M

Suppose $\tilde{\omega}$ is a n-form on an open set U in M with coordinates

$$\{x^1, \dots, x^n\}$$

Since n-forms at $P \in M$ form a 1-D vector space. There exists a function $f(x^1, \dots, x^n)$ such that

$$\tilde{\omega} = f\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n$$

We integrate ove U by first dividing $\mathcal{U}$ into regions (cells) spaned by n-tuples of vectors

$$\{\Delta x^1 \frac{\partial}{\partial x^1}, \Delta x^2 \frac{\partial}{\partial x^2}, \dots, \overset{\text{constant}}{\Delta x^n} \frac{\partial}{\partial x^n}\}$$

where Δx^i's are infinitesimal.

The Integral of f over a region is f multiplied by the following

$$\boxed{\underbrace{\Delta x^1 \, \Delta x^2 \dots \Delta x^n}_{\text{This looks like } dV} = (\tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n)(\Delta x^1 \frac{\partial}{\partial x^1}, \dots, \Delta x^n \frac{\partial}{\partial x^n})}$$

The Integral of f over a cell is written as

$$\int_{\text{cell}} f(x^1, \dots, x^n)\, d^n x \approx \tilde{\omega}$$

locally near p

Add over all the cells and we get the Integral

$$\boxed{\int \overset{\text{n-form}}{\tilde{\omega}} \equiv \int f(x^1, \dots, x^n)\, dx^1 \dots dx^n}$$

Integrating over region of $\mathcal{U}$

we will show this is coordinate independent.

<u>Example</u> In 2D with coordinates (λ, u) the above yields

$$\int \tilde{\omega} = \int f(\lambda, u)\, \tilde{d}\lambda \wedge \tilde{d}u \quad \rightarrow \text{calculus of manifold}$$

$$= \int f(\lambda, u)\, d\lambda\, d\mu \quad \rightarrow \text{calculus of } \mathbb{R}^2$$

Check transformation of coordinate $(\lambda, \mu) \rightarrow (x, y)$

$$\tilde{d}\lambda = \tilde{d}\lambda(x,y) = \frac{\partial \lambda}{\partial x} \tilde{d}x + \frac{\partial \lambda}{\partial y} \tilde{d}y$$

$$\tilde{d}\mu = \frac{\partial \mu}{\partial x} \tilde{d}x + \frac{\partial \mu}{\partial \lambda} \tilde{d}y$$

We build the form:

$$\tilde{d}\lambda \wedge \tilde{d}\mu = \left(\frac{\partial \lambda}{\partial x} \tilde{d}x + \frac{\partial \lambda}{\partial y} \tilde{d}x\right)\left(\frac{\partial \mu}{\partial x} \tilde{d}x + \frac{\partial \mu}{\partial x} \tilde{d}y\right)$$

$$= \frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial x} \overset{\nearrow 0}{\tilde{d}x \wedge \tilde{d}x} + \frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial y} \tilde{d}x \wedge \tilde{d}y + \frac{\partial \lambda}{\partial y} \frac{\partial \mu}{\partial x} \tilde{d}y \wedge \tilde{d}x + \frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial y} \overset{\nearrow 0}{\tilde{d}y \wedge \tilde{d}y}$$

$$= \left(\frac{\partial \lambda}{\partial x} \frac{\partial \mu}{\partial \lambda} - \frac{\partial \lambda}{\partial y} \frac{\partial \mu}{\partial x} \right) \tilde{d}x \wedge \tilde{d}y$$

$$= \frac{\partial(\lambda, \mu)}{\partial(x, y)} \tilde{d}x \wedge \tilde{d}y$$ Jacobian of the transformation

For a n-dimensional manifold we can integrate an n-form to get a non-zero result

For a submanifold of order p, we can integrate a p-form to get a non-zero result

Lec 16 - online

the lectures are online since francis is out of town. Therefore i will not record sound

§ 4.9 N- Vectors, duals and the symbol $\epsilon_{ij\cdots k}$

A completely anti symmetric $\binom{p}{0}$ tensor is a p vector. On a n-dimensional manifold this has dim C^n_p

The following spaces all have the same dimension

$$\left.\begin{array}{l} p\text{- forms} \\ (n-p) \text{ forms} \\ p\text{ - vectors} \\ (n-p) \text{ Vectors} \end{array}\right\} \quad \text{Size } \binom{n}{p} = \binom{n}{n-p}$$

Example
if $n=3$

	$p=1$	$p=2$	$p=3$
dim	3	3	1

inner product

We could use the metric tensor, to map a $\binom{p}{0}$ tensor to a $\binom{0}{p}$ tensor and backwards
It can be shown that since the metric tensor is symmetric, this process preserves anti symmetry

Even without a metric, the volume n-form, $\tilde{\omega}$, yields a mapping from p-vectors to (n-p)-forms
this map is the dual map or the hodge-star map [not mentioned in text book

Suppose T is a 2-vector, with components

$$T^{i\dots k} = T^{[i\dots k]}$$

with $\tilde{\omega}$, we can defin a tensor $\tilde{A}$ such that

$$\boxed{\underbrace{A_{j\dots l}}_{(n-q)\text{ form}} = \frac{1}{q!}\underbrace{\omega_{i\dots k j\dots l}}_{n\text{-form}} \underbrace{T^{i\dots k}}_{q\text{- vector}}} \quad \text{or} \quad \tilde{A} = \tilde{\omega}(T)$$

independent of coordinates

<u>Notation</u>: $\tilde{A} = *T$ and say that $\tilde{A}$ is the dual of T w.r.t. $\tilde{\omega}$

This is invertable and therefore we can bring p- forms to (n-p) vectors

Examples: Consider $\mathbb{E}^3$ in terms of cartesian coordinates

$\bar{U}, \bar{V}$ are both vectors

end of ch 2?

In cartesian coordinates, the elements of the associated 1-forms are equal (b/c of δ)

$$\tilde{U} = g_1(\bar{u}) \text{ or } U_i = g_{ij} U^j = \delta_{ij} U^j$$

and

$$\tilde{V} = g_1(\bar{V}) \text{ or } V_i = g_{ij} V^j = \delta_{ij} V^j$$

Typically, we write these as

basis for 1-forms

$$\tilde{U} = U_1 \tilde{d}x^1 + U_2 \tilde{d}x^2 + U_3 \tilde{d}x^3 = U_i \tilde{d}x^i$$
$$\tilde{V} = V_1 \tilde{d}x^1 + V_2 \tilde{d}x^2 + V_3 \tilde{d}x^3 = V_i \tilde{d}x^i$$

With $\tilde{U}$ and $\tilde{V}$ we can use the wedge to find the following 2-form

$$\tilde{U} \wedge \tilde{V} = (U_1 \tilde{d}x^1 + U_2 \tilde{d}x^2 + U_3 \tilde{d}x^3) \wedge (V_1 \tilde{d}x^1 + V_2 \tilde{d}x^2 + V_3 \tilde{d}x^3)$$

0 b/c symmetry

$$= U_1 V_1 \underbrace{\tilde{d}x^1 \wedge \tilde{d}x^1}_{\text{0 b/c of symmetry}} + U_1 V_2 \tilde{d}x^1 \wedge \tilde{d}x^2 + U_1 V_3 \tilde{d}x^1 \wedge \tilde{d}x^3 + U_2 V_1 \tilde{d}x^2 \wedge \tilde{d}x^1 + U_2 V_2 \tilde{d}x^2 \wedge \tilde{d}x^2 + U_2 V_3 \tilde{d}x^2 \wedge \tilde{d}x^3$$

b/c wedge product is anti-symmetric

0 b/c of symmetry

$$+ U_3 V_1 \tilde{d}x^3 \wedge \tilde{d}x^1 + U_3 V_2 \tilde{d}x^3 \wedge \tilde{d}x^2 + U_3 V_3 \tilde{d}x^3 \wedge \tilde{d}x^3$$

Switching some bases :

$$\tilde{U} \wedge \tilde{V} = (U_1 V_2 - U_2 V_1) \tilde{d}x^1 \wedge \tilde{d}x^2$$
$$+ (U_2 V_3 - U_3 V_2) \tilde{d}x^2 \wedge \tilde{d}x^3$$
$$+ (U_3 V_1 - U_1 V_3) \tilde{d}x^3 \wedge \tilde{d}x^1$$

Coefficents are similar to cross product

Find the dual of this expression.

$$* (\tilde{U} \wedge \tilde{V}) = \frac{1}{2!} \omega_{ijk} (\tilde{U} \wedge \tilde{V})^{ij}$$

Francis claims in lecture, in piazza i'm not sure this is true

$$= \frac{1}{2!} [\omega_{123} (\tilde{U} \wedge \tilde{V})^{12} + \omega_{132} (\tilde{U} \wedge \tilde{V})^{13}$$
$$+ \omega_{231} (\tilde{U} \wedge \tilde{V})^{23} + \omega_{213} (\tilde{U} \wedge \tilde{V})^{21}$$
$$+ \omega_{321} (\tilde{U} \wedge \tilde{V})^{32} + \omega_{312} (\tilde{U} \wedge \tilde{V})^{31}]$$

$$= \frac{1}{2} [(U_1 V_2 - U_2 V_1)^* \tilde{d}x^1 \wedge \tilde{d}x^2 + (U_3 V_1 - U_1 V_3)^* \tilde{d}x^3 \wedge \tilde{d}x^1 + (U_2 V_3 - U_3 V_2)^* \tilde{d}x^2 \wedge \tilde{d}x^3 + (U_1 V_2 - U_2 V_1)^* \tilde{d}x^1 \wedge \tilde{d}x^2$$
$$+ (U_3 V_1 - U_1 V_3)^* \tilde{d}x^3 \wedge \tilde{d}x^1 + (U_2 V_3 - U_3 V_2)^* \tilde{d}x^2 \wedge \tilde{d}x^3]$$

$$= (U_1 V_2 - U_2 V_1) *(\tilde{d}x^1 \wedge \tilde{d}x^2) + (U_2 V_3 - U_3 V_2) *(\tilde{d}x^2 \wedge \tilde{d}x^3) + (U_3 V_1 - U_1 V_3) *(\tilde{d}x^3 \wedge \tilde{d}x^1)$$

How do we find the dual of the 2-forms?

Note that
$$* (\tilde{d}x^1 \wedge \tilde{d}x^2) = \partial/\partial x^3$$
$$* (\tilde{d}x^2 \wedge \tilde{d}x^3) = \partial/\partial x^1$$
$$* (\tilde{d}x^3 \wedge \tilde{d}x^1) = \frac{\partial}{\partial x^2}$$

$\boxed{\text{Dual of a 2 form is a one vector}}$

dual of a 2-form

Check: $* (\tilde{d}x^1 \wedge \tilde{d}x^2) = \frac{\partial}{\partial x^1}$, $*\left(\frac{\partial}{\partial x^1}\right) = \tilde{d}x^2 \wedge \tilde{d}x^3$

Coefficent in x-direction $= 1$

Proof $\frac{1}{1!} \omega_{ijk} T^1 = \omega_{1jk} 1 = A_{jk}$ (2-form)

$\Rightarrow A_{23} = 1$ with basis $\tilde{d}x^2 \wedge \tilde{d}x^3$

or $A_{32} = -1$ with basis $\tilde{d}x^3 \wedge \tilde{d}x^2$

$$*(\tilde{U} \wedge \tilde{V}) = (U_1 V_2 - U_2 V_1)\frac{\partial}{\partial x^3} + (U_2 V_3 - U_3 V_2)\frac{\partial}{\partial x^1} + (U_3 V_1 - U_1 V_3)\frac{\partial}{\partial x^2}$$

Compare with the cross product

$$\bar{U} \times \bar{V} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = \begin{bmatrix} U_2 V_3 - U_3 V_2 \\ U_3 V_1 - U_1 V_3 \\ U_1 V_2 - U_2 V_1 \end{bmatrix}$$

Looks Like
$\partial/\partial x^1$
$\partial/\partial x^2$
$\partial/\partial x^3$

$\therefore \; *(\tilde{U} \wedge \tilde{V}) = (\bar{U} \times \bar{V}) \Leftrightarrow (\tilde{U} \wedge \tilde{V}) = *(\bar{U} \times \bar{V})$

This result is unique to $\mathbb{R}^3$.

The map betwe T and $*T$ is invertiable.

Levi - Civita Symbols

$$\epsilon_{ij\ldots k} = \epsilon^{ij\ldots k} = \begin{cases} +1 & \text{if } ij\ldots k \text{ is an even permutation} \\ -1 & \text{if } ij\ldots k \text{ is an odd permutation} \\ 0 & \text{otherwize} \end{cases}$$

example: $\bar{U} \times \bar{V} = \epsilon_{ijk} U^j V^k$

B the differential Calculus of form and it applications

Single variable Calculus States

$$\int_a^b df = f(b) - f(a)$$

We want to derive a derivative operator that reduces to this in the simple case, but is more general

§4.14 The exterior derivative

If M is a 1D manifold, $\tilde{d}$: 0-Form $\rightarrow$ 1-form. It will agree withe the above
takes a function

If $\tilde{\alpha}$ is a p-fom a $\tilde{\beta}$, $\tilde{\gamma}$ are q-forms we requre

(1) $\tilde{d}(\tilde{\beta} + \tilde{\gamma}) = (\tilde{d}\tilde{\beta}) + (\tilde{d}\tilde{\gamma})$ Distributive

(2) $\tilde{d}(\tilde{\alpha} \wedge \tilde{\beta}) = (\tilde{d}\tilde{\alpha}) \wedge \tilde{\beta} + (-1)^p \tilde{\alpha} \wedge (\tilde{d}\tilde{\beta})$ Anti derivation

(3) $\tilde{d}(\tilde{d}\tilde{\alpha}) = 0$

These 3 properties uniquely define $\tilde{d}$.

$\tilde{d}$ is called the exterior derivative

Property (ii) is almost Leibniz but there is an extra $(-1)^p$ to bring $\tilde{d}$ across the p-form

Property (iii) Seems odd but is essential

ex) if $\tilde{\alpha}$ is a function then $\tilde{d}f$ is a one form with component $\frac{\partial f}{\partial x^i}$

$\tilde{d}(\tilde{d}f)$ has components of the form $\frac{\partial^2 f}{\partial x^j \partial x^i}$. But this must be a 2-form and since it must be anti symmetric, it must be 0

§ 4.15 Notation for derivatives

Partial derivatives : $\frac{\partial f}{\partial x^i} = f_{,i}$ 1st derivative

$\frac{\partial V^i{}_j}{\partial x^k}$ $V^i{}_{j,k}$ 1st derivative of a $\binom{1}{1}$ tensor

$\frac{\partial^2 f}{\partial x^k \partial x^i} = f_{,ik}$ 2nd Derivative

Recall that a partial derivative is not a tensor operation in general [which assignment went over this?]

example ① $V^i{}_{j,k}$ need not be $\Lambda^i{}_{a'}\Lambda^{b'}{}_j\Lambda^{c'}{}_k A^{a'}{}_{b',c'}$ What is it?

this part breaks down ... is it coordinate depeneded

② $\tilde{d}f = f_{,i}$ (1-form) and is a tensor operation

③ $[\bar{u},\bar{v}]^i = u^j v^i{}_{,j} - v^j u^i{}_{,j}$ → Lie bracket this is a tensor operator

Each term on the RHS is not a tensor operator but the whole RHS is

Excerse 4.14 on an assigment

a) $\tilde{d}(f\tilde{d}g) = \tilde{d}f \wedge \tilde{d}g$ [with 3rd property means $(-1)^p \tilde{d}\tilde{d}g = 0$]

b) If $\tilde{\alpha} = \frac{1}{p!}\alpha_{i\ldots j}\,\tilde{d}x^i \wedge \cdots \wedge \tilde{d}x^j$ is a p-form

the $\tilde{d}\tilde{\alpha} = \frac{1}{p!}\frac{\partial}{\partial x^k}(\alpha_{i\ldots j})\,\tilde{d}x^k \wedge \tilde{d}x^i \wedge \ldots \wedge \tilde{d}x^j$

and

$$(\tilde{d}\alpha)_{ki\ldots j} = (p+1)\frac{\partial}{\partial x}[k\alpha_{i\ldots j}]$$

or $(\tilde{d}\tilde{\alpha})_{ki\ldots j} = (p+1)\,\alpha_{[i\ldots j,k]}$

Lec 17 - Online

§ 4.16 Familiar examples of exterior derivatives

We can revisit some old friends with a new perspective

① $\tilde{d}$ of a 1-form $\tilde{a}$ in 3D:

components

$\tilde{d}\tilde{a} = \tilde{d}(a_1 \tilde{d}x^1 + a_2 \tilde{d}x^2 + a_3 \tilde{d}x^3)$ two-form

one-form

$= a_{1,j} \tilde{d}x^j \wedge \tilde{d}x^1 + a_{2,j} \tilde{d}x^j \wedge \tilde{d}x^2 + a_{3,j} \tilde{d}x^j \wedge \tilde{d}x^3$ (new index)

all the components that are not zero due to anti symmetry

$= a_{1,2} \tilde{d}x^2 \wedge \tilde{d}x^1 + a_{1,3} \tilde{d}x^3 \wedge \tilde{d}x^1 + a_{2,1} \tilde{d}x^1 \wedge \tilde{d}x^2 + a_{2,3} \tilde{d}x^3 \wedge \tilde{d}x^2 + a_{3,1} \tilde{d}x^1 \wedge \tilde{d}x^3 + a_{3,2} \tilde{d}x^2 \wedge \tilde{d}x^3$

$= (a_{3,2} - a_{2,3}) \tilde{d}x^2 \wedge \tilde{d}x^3 + (a_{1,3} - a_{3,1}) \tilde{d}x^3 \wedge \tilde{d}x^1 + (a_{2,1} - a_{1,2}) \tilde{d}x^1 \wedge \tilde{d}x^2 \rightarrow$ combing like terms by switching the two bases in the wedge

Consider the dual, or hodge-star

$$* \tilde{d}\tilde{a} = (a_{3,2} - a_{2,3}) * (\tilde{d}x^2 \wedge \tilde{d}x^3) + (a_{1,3} - a_{3,1}) * (\tilde{d}x^3 \wedge \tilde{d}x^1) + (a_{2,1} - a_{1,2}) * (\tilde{d}x^1 \wedge \tilde{d}x^2)$$

From before $*(\tilde{d}x^2 \wedge \tilde{d}x^3) = \frac{\partial}{\partial x^1}$ $\quad *(\tilde{d}x^3 \wedge \tilde{d}x^1) = \frac{\partial}{\partial x^2}$ $\quad *(\tilde{d}x^1 \wedge \tilde{d}x^2) = \frac{\partial}{\partial x^3}$

Hence, $* \tilde{d}\tilde{a} = (a_{3,2} - a_{2,3}) \frac{\partial}{\partial x^1} + (a_{1,3} - a_{3,1}) \frac{\partial}{\partial x^2} + (a_{2,1} - a_{1,2}) \frac{\partial}{\partial x^3}$ → Looks like the curl

or

$$* \tilde{d}\tilde{a} = \epsilon_{ijk} \frac{\partial}{\partial x^j} a_k = \epsilon_{ijk} a_{k,j}$$

The RHS is the curl since $\bar{\nabla} \times \bar{a} = \epsilon_{ijk} a_{k,j}$

Summary $* \tilde{d} =$ curl in 3D when applied to 1-forms, when exterior derivative is applied to a scalar it gives the gradient, as seen in ③

② First take the dual of a vector then $\tilde{d}$

Suppose $\bar{a}$ is a vector field

$$*(\bar{a}) = *(a^1 \frac{\partial}{\partial x^1} + a^2 \frac{\partial}{\partial x^2} + a^3 \frac{\partial}{\partial x^3}) = a^1 * (\frac{\partial}{\partial x^1}) + a^2 * (\frac{\partial}{\partial x^2}) + a^3 * (\frac{\partial}{\partial x^3})$$

$$= a^1 (\tilde{d}x^2 \wedge \tilde{d}x^3) + a^2 (\tilde{d}x^3 \wedge \tilde{d}x^1) + a^3 (\tilde{d}x^1 \wedge \tilde{d}x^2)$$

Apply $\tilde{d}$ — Only 1 value for each j that would make the wedges non-zero

$$\tilde{d}(*\bar{a}) = a^1{}_{,j} (\tilde{d}x^j \wedge \tilde{d}x^2 \wedge \tilde{d}x^3) + a^2{}_{,j} (\tilde{d}x^j \wedge \tilde{d}x^3 \wedge \tilde{d}x^1) + a^3{}_{,j} (\tilde{d}x^j \wedge \tilde{d}x^1 \wedge \tilde{d}x^2)$$

($j=1$, $j=2$, $j=3$)

$$= (a^1{}_{,1} + a^2{}_{,2} + a^3{}_{,3}) \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \tilde{d}x^3$$

Note $\tilde{d} * \tilde{a} = (\bar{\nabla} \cdot \bar{a}) \tilde{\omega}$ (divergence)

③ $\tilde{d}f = f_{,i} \tilde{d}x^i$ (gradient)

§4.17 Integrability conditions for PDEs

Consider the system of 2 PDEs:

$$\frac{\partial f}{\partial x} = g(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y} = h(x,y)$$

More interested in when a solution exists

Let (x,y) be coordinates of a manifold. Further, define $\underbrace{a_x = g \text{ and } a_y = h}_{\text{vector components}}$ and then we can re write the system compactly as

$$\boxed{f_{,i} = a_i} \quad i = 1,2$$

one form notation

A coordinate-independent version of this is

$$\tilde{d}f = \tilde{a}$$

→ why $\tilde{a}$? i thought a_x, a_y was a vector

This holds because $\frac{\partial f}{\partial x}\tilde{d}x + \frac{\partial f}{\partial y}\tilde{d}y = g\tilde{d}x + h\tilde{d}y$

If f is a soln then it must follow that

$$\tilde{d}(\tilde{d}f) = \tilde{d}\tilde{a} = 0$$

In component form this becomes

$$\tilde{d}\tilde{a} = \tilde{d}(g\tilde{d}x + h\tilde{d}y)$$

$$= \frac{\partial g}{\partial y}\tilde{d}y \wedge \tilde{d}x + \frac{\partial h}{\partial x}\tilde{d}x \wedge \tilde{d}y$$

→ $\frac{\partial g}{\partial x} = 0$ b/c of anti-symmetry

$$= \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right)\tilde{d}x \wedge \tilde{d}y = 0$$

→ Greens theorem?

or

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$a_{2,1} - a_{1,2} = 0 \quad \text{or} \quad a_{[i,j]} = 0$$

→ where does this notation come from?

This condition is necessary and later we will show it is sufficent for a solution to exist

§4.18 Exact forms

Observe, if $\tilde{\alpha} = \tilde{d}\tilde{\beta}$ the $\tilde{d}\tilde{\alpha} = \tilde{d}(\tilde{d}\beta) = 0$

If $\tilde{d}\tilde{\alpha} = 0$ then $\tilde{\alpha}$ is closed

If $\tilde{\alpha} = \tilde{d}\tilde{\beta}$ then $\tilde{\alpha}$ is exact

Clearly an exact form is closed. It can be shown that any closed form is exact

§ 4.20 Lie derivatives of forms

If $\tilde{\omega}$ is a p-form then

$$\underbrace{\mathcal{L}_{\bar{V}}\tilde{\omega}}_{\text{p-form}} = \underbrace{\tilde{d}[\underbrace{\tilde{\omega}(\bar{V})}_{\text{p-1 form}}]}_{\text{p-form}} + \underbrace{(\tilde{d}\underbrace{\tilde{\omega}}_{\text{p-1 form}})(\bar{V})}_{\text{p-form}}$$

Idea of proof:

case 1 $\tilde{\omega}$ is a 0-form, $\tilde{\omega} = f$

LHS $\mathcal{L}_{\bar{V}}\tilde{\omega} = \mathcal{L}_{\bar{V}} f = \bar{V}(f) = \frac{df}{d\lambda}$

(RHS1) does not make sense, this term is ignored   we cant have a function acting on a vector field

(RHS2) $(\tilde{d}\tilde{\omega})(\bar{V}) = (\tilde{d}f)(\bar{V}) = \frac{\partial f}{\partial x^i}\tilde{d}x^i\left(V^j \frac{\partial}{\partial x^j}\right)$

$$(\tilde{d}\tilde{\omega})(\bar{V}) = V^j f_{,i}\underbrace{\tilde{d}x^i\left(\frac{\partial}{\partial x^j}\right)}_{=\delta^i_j}$$

$$= V^i f_{,i} = \frac{df}{d\lambda}$$

Case 2: $\tilde{\omega}$ is a 1-form $\tilde{\omega} = \omega_i \tilde{d}x^i$ [one form notation]

(RHS1) $\tilde{d}(\tilde{\omega}(\bar{V})) = \tilde{d}(\omega_i V^i) = (\omega_i V^i)_{,j}\tilde{d}x^j$

(RHS2) $(\tilde{d}\tilde{\omega})(\bar{V}) = (\tilde{d}(\omega_i \tilde{d}x^i))(\bar{V})$

$= (\omega_{i,j}\tilde{d}x^j \wedge \tilde{d}x^i)(\bar{V})$ → which formula does this use?

→ new index for V

$= \omega_{i,j}(\tilde{d}x^j \otimes \tilde{d}x^i - \tilde{d}x^i \otimes \tilde{d}x^j)\left(V^k \frac{\partial}{\partial x^k}\right)$ → translation of $\wedge$ to outer product

$= \omega_{i,j}\left(\tilde{d}x^j\left(V^k \frac{\partial}{\partial x^k}\right) \otimes \tilde{d}x^i - \tilde{d}x^i\left(V^k\frac{\partial}{\partial x^k} \otimes \tilde{d}x^j\right)\right)$ → distributing $V^k\frac{\partial}{\partial x^k}$

$\quad = \delta^j_k \qquad\qquad = \delta^i_k$ → where did the cross go?

$= \omega_{i,j}(V^j\tilde{d}x^i - V^i\tilde{d}x^j)$

RHS $= (\omega_i V^i)_{,j}\tilde{d}x^j + \omega_{i,j}(V^j\tilde{d}x^i - V^i\tilde{d}x^j)$

product rule

$= \cancel{\omega_{i,j}V^i\tilde{d}x^j} + \omega_i V^i_{,j}\tilde{d}x^j + \omega_{i,j}(V^j\tilde{d}x^i - \cancel{V^i\tilde{d}x^j})$

$= \omega_{i,j}V^j\tilde{d}x^i + \omega_j V^j_{,i}\tilde{d}x^i$

$= (\omega_{i,j}V^j + \omega_j V^j_{,i})\tilde{d}x^i = (\mathcal{L}_{\bar{V}}\tilde{\omega})_i$

Can fully prove using induction

Lec 18 - Nov 14

§4.21 Lie derivatives and Exterior derivative commute

Woohoo Stokes thereom!!!

Thm: $\mathcal{L}_{\vec{V}}$ and $\tilde{d}$ commute

We will prove this for a one-form. or n-form?

proof: we need a formula from § 4.20

$$\mathcal{L}_{\vec{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\vec{V})] + (\tilde{d}\tilde{\omega})(\vec{V}) \Big\} \to \text{Still have n-form}$$

Gives tensor of same space?

but replace $\tilde{\omega}$ with $\tilde{d}\tilde{\omega}$ → true for n+1, exact form

this in not zero b/c of how $[\tilde{d}\tilde{\omega}(\vec{V})]$ gets calculated

B/c of properties of $\tilde{d}$

$$\mathcal{L}_{\vec{V}}\tilde{d}\tilde{\omega} = \tilde{d}[\tilde{d}\tilde{\omega}(\vec{V})] + \cancel{(\tilde{d}(\tilde{d}\tilde{\omega})}(\vec{V})$$

From the first equation we know

$$\tilde{d}\tilde{\omega}(\vec{V}) = \mathcal{L}_{\vec{V}}\tilde{\omega} - \tilde{d}[\tilde{\omega}(\vec{V})]$$

Substitute into the RHS of the provious equation

$$\mathcal{L}_{\vec{V}}\tilde{d}\tilde{\omega} = \tilde{d}[\mathcal{L}_{\vec{V}}\tilde{\omega} - \cancel{\tilde{d}[\tilde{\omega}(\vec{V})]}]$$

→ the [ ] around $\tilde{\omega}(\vec{V})$ make an n-1 form, hence $\tilde{d}$ is acting on an n-1 form and $\tilde{d}\tilde{d}(\tilde{\omega}(V))$ is zero

$$= \tilde{d}\mathcal{L}_{\vec{V}}\tilde{\omega} \quad \therefore \text{commutes}$$

§4.22 Stokes thm

We show that the exterior derivative is the inverse of integration, in particular

$$\int_U \tilde{d}\tilde{\omega} = \int_{\partial U}\tilde{\omega}$$ Different from $\tilde{\omega}$ we use

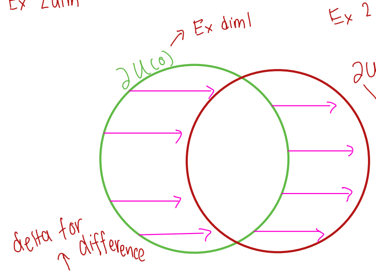
We can integrate n-forms over n-dimension. and we can integrate n-1 forms over n-1 dimenson.

If U is n-dimensionl then its boundary is n-1 dimensonial. The boundary is the exterior of U, and why this is called exterior calculus

Assume, U is a smooth, orentable volume on M that is connected. then ∂U is a submanifold of M. also $\vec{\xi}$ is a vector Field on M

Suppose $\mathcal{U} = \mathcal{U}(0)$ is a region on M with boundary $\partial\mathcal{U} = \partial\mathcal{U}(0)$, and $\mathcal{U}(\epsilon)$ is the Lie dragged region along $\vec{\xi}$

for Ex 2dim — Ex 2 dim — Ex 1 dim — Ex. 2 dim



$\partial\mathcal{U}(\epsilon) \rightarrow$ Ex dim1

partial to denote boundary

BIRDS EYE VIEW

$\delta\mathcal{U}(\epsilon)$ is the area between $\mathcal{U}(0)$ and $\mathcal{U}(\epsilon)$, which is dim 2

or $\delta\mathcal{U}(\epsilon) \equiv \mathcal{U}(\epsilon) - \mathcal{U}(0)$

To find the change of an integral from $\mathcal{U}(0)$ to $\mathcal{U}(\epsilon)$ we compute the following

$$\int_{\mathcal{U}(\epsilon)} \tilde{\omega} - \int_{\mathcal{U}(0)} \tilde{\omega} = \int_{\delta\mathcal{U}(\epsilon)} \tilde{\omega} \quad *$$

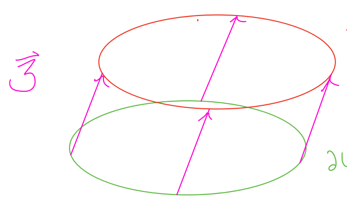
n-form

$\tilde{\omega}$ is an n-form, the n-form that gives us volume. Anti-symmetric and $\binom{0}{N}$ tensor

$\rightarrow$ NO x^1!!

Suppose V is a coordinate patch of $\partial\mathcal{U}(0)$ with coordinates $\{x^2, \ldots, x^n\}$

If we Lie drag $\partial\mathcal{U}(0)$ along $\vec{\xi}$ a distance of ϵ, then we have coordinates $\{x^1 = \epsilon, x^2, \ldots, x^n\}$



SIDE PROFILE

This part has a patch, the $\epsilon = x^1$ coordinate patch gives us a way to describe the boundary of the cylinder

$\vec{\xi}$ is no where tangent to $\partial\mathcal{U}(0)$

We investigate the integral of $\delta V(\epsilon)$ and then extend this to $\delta\mathcal{U}(\epsilon)$

We introduce, $\boxed{\tilde{\omega} = f(x^1, \ldots, x^n)\, \tilde{d}x^1 \wedge \ldots \wedge \tilde{d}x^n}$

If $\epsilon \ll 1$, then the integral over $\delta V(\epsilon)$ is,

This is like Riemann sums

$$\int_{\partial V(\epsilon)} \tilde{\omega} = \int_{V(0)} \left[\int_0^{\epsilon} f\, dx^1 \right] dx^2 \ldots dx^n$$

$$= \epsilon \int_{V(0)} f(0, x^2, \ldots, x^n)\, dx^2 \ldots dx^n + O(\epsilon) \longrightarrow \text{Linearizing since } \epsilon \ll 1$$

smaller than ϵ

$$= \epsilon \int_{V(0)} \tilde{\omega}(\vec{\xi}) \Big|_{\partial\mathcal{U}(0)} + o(\epsilon)$$

This is a change to Calculus on manifold

Note: $\xi = \frac{\partial}{\partial x^1}$ by design

$$\tilde{\omega}\left(\frac{\partial}{\partial x^1}\right) = f(x^1, \dots, x^n)\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n \left(\frac{\partial}{\partial x^1}\right)$$

$$= f(0, x^2, \dots, x^n)\, \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

and $$\int_0^\epsilon f\, dx^1 \approx \epsilon\, f(0, x^2, \dots, x^n) + o(\epsilon)$$

Summary, $$\int_{\partial V(\epsilon)} \tilde{\omega} = \epsilon \int_{V(0)} \tilde{\omega}(\xi)\Big|_{\partial u} + o(\epsilon) \quad ** $$

Now consider,

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left[\int_{U(\epsilon)} \tilde{\omega} - \int_{U(0)} \tilde{\omega}\right]$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\delta U(\epsilon)} \tilde{\omega}$$ From first eqn *

If $\delta U(\epsilon) \approx \delta V(\epsilon)$, which is true if $\epsilon \ll 1$, then

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\epsilon \int_{V(0)} \tilde{\omega}(\xi)\Big|_{\partial u} + o(\epsilon)\right)$$

$$\boxed{\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \int_{\partial U(0)} \tilde{\omega}(\xi)\Big|_{\partial U(0)}}$$

But using linerization we can approximate the integrand on the LHS using

$$\tilde{\omega} = \epsilon \mathcal{L}_\xi \tilde{\omega} + o(\epsilon)$$? Francis doest get where this comes from

$\to$ on $U(0)$

Sub into LHS

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \frac{d}{d\epsilon} \int_{U(0)} \epsilon \mathcal{L}_\xi \tilde{\omega} + o(\epsilon)$$

$$\frac{d}{d\epsilon} \int_{U(\epsilon)} \tilde{\omega} = \int_{U(0)} \mathcal{L}_\xi \tilde{\omega}$$

But the formula for $\mathcal{L}_\xi \tilde{\omega}$ yields

$$\int_{U(0)} \mathcal{L}_\xi \tilde{\omega} = \int_{U(0)} \tilde{d}[\tilde{\omega}(\xi)] + (\tilde{d}(\tilde{\omega}))(\xi)$$ this is an n+1 form an hence $\to 0$

$$= \int_{U(0)} \tilde{d}[\tilde{\omega}(\xi)]$$

If we combine our formula's we get

$$\frac{d}{dt}\int_{U(t)} \tilde{\omega} = \int_{U(0)} \tilde{\partial}\left[\tilde{\omega}(\bar{\xi})\right] = \int_{\partial U(0)} \tilde{\omega}\,(\bar{\xi})$$

From above

befor formula with $\mathcal{L}$

Define $\tilde{\alpha} = \tilde{\omega}(\bar{\xi}) = i_{\xi}\,\tilde{\omega}$ and get

$$\boxed{\int_{U(0)} \tilde{\partial}\,\tilde{\alpha} = \int_{\partial U(0)} \tilde{\alpha}}$$

Stokes theorem

Lec 19 ~ Nov 16^{th}

PHYSICS NEXT WEEK

§ 4.22 Stokes theorem

Recall $$\frac{d}{d\epsilon}\int_{U(\epsilon)} \tilde{\omega} = \int_{U(0)} \mathcal{L}_{\vec{\xi}}\, \tilde{\omega}$$

Justification:

LHS: We integrate $\tilde{\omega}$ (n-form) over the n-volume $U(\epsilon)$

To obtain the RHS we taylor expand $\tilde{\omega}$ at $U(\epsilon)$ around $U(0)$

little o

Aside: $f(\vec{x}_0 + \epsilon\vec{\xi}) = f(\vec{x}_0) + \epsilon\vec{\xi}\cdot\vec{\nabla} f(\vec{x}_0) + o(\epsilon)$

Use this idea to appox $\tilde{\omega}$,

Since the Lie derivative is the directional derivative

$$\tilde{\omega}\Big|_{U(\epsilon)} = \tilde{\omega}\Big|_{U(0)} + \epsilon\, \mathcal{L}_{\vec{\xi}}\, \tilde{\omega}\Big|_{U(0)} + o(\epsilon)$$

If you plug this into the LHS,

$$\frac{d}{d\epsilon}\int_{U(\epsilon)} \tilde{\omega} = \frac{d}{d\epsilon}\int_{U(0)} \tilde{\omega} + \epsilon\, \mathcal{L}_{\vec{\xi}}\, \tilde{\omega} + o(\epsilon)$$

$$= \int_{U(0)} \mathcal{L}_{\vec{\xi}}\, \tilde{\omega} + \text{small stuff}$$

can convert to vector

Example: In $\mathbb{E}^2$ consider $\tilde{\alpha} = \alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x^2$

we apply $\tilde{d}$ and obtain

$$\tilde{d}\alpha = \tilde{d}(\alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x)$$ is a product rule going on [Next assignment, comes out friday Nov 17)

$$= \alpha_{1,2}\, \tilde{d}x^2 \wedge \tilde{d}x^1 + \alpha_{2,1}\, \tilde{d}x^1 \wedge \tilde{d}x^2$$

$$= \alpha_{2,1} - \alpha_{1,2}\, \tilde{d}x^1 \wedge \tilde{d}x^2$$

Stokes' theorem can be written as

$$\int_U \tilde{d}\tilde{\alpha} = \int_{\partial U} \tilde{\alpha}$$

$$\int_U (\alpha_{2,1} - \alpha_{1,2})\, \tilde{d}x^1 \wedge \tilde{d}x^2 = \int_{\partial U} \overbrace{\alpha_1 \tilde{d}x^1 + \alpha_2 \tilde{d}x^2}^{(\alpha_1,\alpha_2)\cdot(\tilde{d}x^1,\tilde{d}x^2)}$$

We can rewrite this in terms of "regular" integrals

$$\int_u \left(\frac{\partial \alpha_2}{\partial x} - \frac{\partial \alpha_1}{\partial y}\right) dx\, dy = \oint (\alpha_1, \alpha_2) \cdot \left(\frac{dx}{d\lambda}, \frac{dy}{d\lambda}\right) d\lambda$$

$$= \oint_{\partial u} \bar{\alpha} \cdot d\bar{x}$$

This is greens theorem, a special case of Stokes thm

§4.23 Gauss' theorem and the defn of divergence

Recall Stokes thm can be written a

not really need

$$\int_u \tilde{d}\tilde{\alpha} = \int_{\partial u} \tilde{\alpha} \quad \text{or} \quad \boxed{\int_u \tilde{d}[\tilde{\omega}(\bar{\xi})] = \int_{\partial u} \tilde{\omega}(\bar{\xi})\Big|_{\partial u}} \quad *$$

Suppose $\tilde{\omega} = \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$

is it because $\frac{\partial}{\partial x^1} \wedge \tilde{d}x^1 = 0$

then $\tilde{\omega}(\bar{\xi}) = \xi^1 \tilde{d}x^2 \wedge \dots \wedge \tilde{d}^n - \xi^2 \tilde{d}x^1 \wedge \tilde{d}x^3 \wedge \dots \tilde{d}x^n + \dots$

where does the negative come from?

We compute $\tilde{d}$ of the above and get

$$\tilde{d}[\tilde{\omega}(\bar{\xi})] = \underline{\xi^1{}_{,1}}\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n + \underline{\xi^2{}_{,2}}\, \tilde{d}x^1 \wedge \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n + \dots + \xi^n{}_{,n}\, \tilde{d}x^1 \wedge \dots \wedge \tilde{d}x^n$$

$$\Rightarrow \tilde{d}[\tilde{\omega}(\bar{\xi})] = \xi^i{}_{,i}\, \tilde{\omega}$$

We define the $\tilde{\omega}$-divergence of $\bar{\xi}$ as $\boxed{(\operatorname{div}_{\tilde{\omega}} \bar{\xi})\, \tilde{\omega} \equiv \tilde{d}[\tilde{\omega}(\bar{\xi})]}$

If we use components such that ∂U is a surface of constant x^1, then the restriction of $\tilde{\omega}(\bar{\xi})$ to ∂U is

$$\tilde{\omega}(\bar{\xi})\Big|_{\partial u} = \xi^1 \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

or

$$= \tilde{d}x^1(\bar{\xi})\, \tilde{d}x^2 \wedge \dots \wedge \tilde{d}x^n$$

In general if $\tilde{n}$ is a 1-form normal to the boundary of U (∂U), which means that $\tilde{n}(\bar{\eta}) = 0 \ \forall\, \bar{\eta}$ tangent to ∂U

and if $\tilde{\sigma}$ is an $(n-1)$-form with

$$\tilde{\omega} = \tilde{n} \wedge \tilde{\sigma}$$

then

$$\tilde{\omega}(\bar{\xi})\Big|_{\partial u} = \tilde{n}(\bar{\xi})\, \tilde{\sigma}\Big|_{\partial u}$$

Therefore the original form of Stokes thm * becomes

$$\int_{\mathcal{U}} (\mathrm{div}_{\tilde{\omega}} \vec{\xi}) \tilde{\omega} = \int_{\partial \mathcal{U}} \tilde{n}(\vec{\xi}) \tilde{\sigma}$$

with $\tilde{\sigma}$ restricted to $\partial \mathcal{U}$ and $\tilde{n} \wedge \tilde{\sigma} = \tilde{\omega}$

In component form, this becomes

$$\boxed{\int_{\mathcal{U}} \xi^i{}_{;i}\, d^n x = \oint_{\partial \mathcal{U}} \xi^i n_i\, d^{n-1} x}$$

Gauss' divergence thereom in $\mathbb{R}^n$

§ 4.25 Differential forms and Differential Equations

Consider the DE $\frac{dy}{dx} = f(x,y)$

we often rewrite it as $dy = f(x,y)\,dx$

whats the connection between the two?

If M is a 2D manifold with coordinates (x,y), then we consider the following

$$\tilde{d}y - f(x,y)\tilde{d}x = 0$$ → this is inspiration

where f is a function on M.

Suppose $\bar{V}$ is a vector at $P \in M$ with components $(1, f(P))$

Consider $\tilde{d}y(\bar{V}) = \tilde{d}y(1, f(P)) = f(P)$

$\bar{V} = \frac{\partial}{\partial x} + f(P)\frac{\partial}{\partial y}$

$\tilde{d}x(\bar{V}) = \tilde{d}x(1, f(P)) = 1$

This implies, $\boxed{(\tilde{d}y - f\tilde{d}x)(\bar{V}) = 0}$ $= \tilde{d}y(\bar{V}) - f\tilde{d}(x)(\bar{V}) = f(P) - f = 0$

Solns to the DEs define a submanifold of M whose tangent <u>annul</u> the 1-form

send to 0

Submanifolds that annul the 1-form are solutions to this can be generalized to n-forms with Frobenius thm

Question Given a DE, what are the equivalent form?

example $\frac{dx^2}{dt^2} + \omega_0^2 x =$ ω_0 is constant. Harmonic oscillator

or $\frac{dx}{dt} = \omega_0 y$ and $\frac{dy}{dt} = -\omega_0 x$ System of 1st Order Equations or $\frac{dx}{dt} - \omega_0 y = 0$ and $\frac{dy}{dt} + \omega_0 x = 0$

The 1-forms to consider are:

$$\tilde{\alpha} = \tilde{d}x - \omega_0 y \tilde{d}t$$

$$\tilde{\beta} = \tilde{d}y + \omega_0 x \tilde{d}t$$

Finding Submanifolds that annul these forms is equivalent to Solving DEs.

The manifold is 3D with coordinates $[x, y, z]$ and the Solution is 1D.

§ 4.26 Frobenius' theorem (differential forms version)

The set of forms $\{\tilde{\beta}_i\}$ at $P \in M$ define a Subspace of Vectors, $T_pS \subset T_pM$, each of which annuls $\tilde{\beta}_i$, i.e. For all $\bar{V} \in T_pS$, $\tilde{\beta}_i(\bar{V}) = 0 \quad \forall i = 1, \dots, n$

The Set T_pS is called annihilator of $\{\tilde{\beta}_i\}$

The complete ideal consists of all the forms at P whose restriction to T_pS vannishes

Note: if $\tilde{\gamma}$ is a form at P then $\tilde{\gamma} \wedge \tilde{\beta}_i$ is 0 when restricted to T_pS and therefore $\tilde{\gamma} \wedge \tilde{\beta}_i$ is in the Complete ideal

A complete ideal has a basis $\{\tilde{\alpha}_j\}$ that generates the ideal ie

the complete ideal of $\{\tilde{\alpha}_j\}$ is the Same as the Complete ideal of $\{\tilde{\beta}_j\}$

All of this extends from vectors to vector fields

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§ 4.26 Frobenius theorem.

$\{\tilde{\beta}_i\}$ defines a subspace, $T_pS \subset T_pM$, each of which annuls $\tilde{\beta}_i$.

$$\forall \bar{V} \in T_pS \quad \text{then} \quad \tilde{\beta}_i(\bar{V}) = 0 \qquad \forall i=1,\dots,n$$

T_pS is the annihilator of $\{\tilde{\beta}_i\}$.

The complete ideal consists of all the forms whos restriction to T_pS vanishes.

Note : If $\tilde{\gamma}$ is a form then $\tilde{\gamma} \wedge \tilde{\beta}_i$ is 0 when restricted to T_pS $\therefore$ $\tilde{\gamma}\wedge\tilde{\beta}_i$ is in the complete ideal ★

$\{\tilde{\alpha}_j\}$ is closed if each $\tilde{d}\tilde{\alpha}_j$ is in the complete ideal generated by $\{\tilde{\alpha}_j\}$

Aside: A complete ideal has a basis $\{\tilde{\alpha}_j\}$ that generates ideal

Frobenius Theorem:

Suppose $\{\tilde{\alpha}_i, i=1,\dots,m\}$ is a linearly independent set of 1-form fields in an open set $U \subset M$, where M is an n-dimensional manifold. The set $\{\tilde{\alpha}_i\}$ is closed iff functions $\{P_{ij}, Q_j \; i,j=1\dots m\}$ such that

an array w 2 index

a 1- array

$$\tilde{\alpha}_i = \sum_{j=1}^{m} P_{ij}\,\tilde{d}Q_j$$

Idea: In general to Solve DEs, we want to find Solutions to $\{\tilde{\alpha}_i = 0\}$. The Solution to this set of equations by $Q_j =$ constant

This set of Q_j are solution to the equations $\{\tilde{\alpha}_i = 0\}$ and Each Q_j defines an m-dimensional Submanifold of M. and its tangent Vectors annul $\{\tilde{d}Q_j\}$ and also $\{\tilde{\alpha}_j\}$

example: Suppose $\tilde{\alpha} = \tilde{d}f$ this satisfieds the above with $P_{11} = 1$ and $f = Q$. f exists iff $\tilde{d}\tilde{\alpha} = 0$

exercise 4.30 $\{\tilde{\alpha}_j, j=1\dots m\}$ is a linearly independent set of 1-forms then any form $\tilde{\gamma}$ is in the complete ideal iff $\tilde{\gamma}\wedge\tilde{\alpha}_1\wedge\tilde{\alpha}_2\wedge\dots\wedge\tilde{\alpha}_m = 0$

§ 5 Applications to Physics

§ 5 A Thermodynamics

§ 5.1 Simple Systems

Consider a one-component fluid where the conservation of energy dictates that

path independent?

$$\boxed{\delta Q = P\,\delta V + dU}$$

1st Law of Thermodynamics

variation (changes?) path dependent?

where U is the internal energy

δQ heat absorbed

$P\delta V$ work done by the fluid

P, V pressure and Volume

This law can be written in terms of 1-forms on a 2D manifold with coordinates (V, U) assuming good coordinates

Then $P(V, U)$ is a function on M that is the equation of State

internal energy

volume

On the RHS, it would make sense to write it as

$$P\tilde{\partial} V + \underbrace{\tilde{\partial} U}_{\text{exact}}$$

Since $\tilde{\partial} V$ and $\tilde{\partial} U$ are one forms, we deduce that the LHS $\tilde{\delta} Q$ is a 1-form as well.

may not be

Question is $\tilde{\delta} Q = \tilde{\partial} Q$, is it an exact one-form? If yes, then $\tilde{\partial}\tilde{\partial} Q = 0$ and we deduce

$$0 = \tilde{\partial}(\tilde{\partial} Q) = \tilde{\partial}(P\tilde{\partial} V + \underbrace{\tilde{\partial} U}_{0})$$

Re look up gradient of function again!

gradient of P

That reduces to

$$0 = \tilde{\partial}(P\tilde{\partial} V) = \boxed{\tilde{\partial} P \wedge \tilde{\partial} V}$$ Assignment 5.

$$\tilde{\partial} P \wedge \tilde{\partial} V + \underbrace{P\,\tilde{\partial}\tilde{\partial} V}_{\to 0}$$

we can simplify this,

$$\left(\left(\frac{\partial P}{\partial V}\right)_U \tilde{\partial} V + \left(\frac{\partial P}{\partial U}\right)_V \tilde{\partial} U\right) \wedge \tilde{\partial} V = 0$$

$$\tilde{\partial} V \wedge \tilde{\partial} V = 0$$

$$\left(\frac{\partial P}{\partial U}\right)_V \tilde{\partial} U \wedge \tilde{\partial} V = 0$$

But this can only be true if $\left(\frac{\partial P}{\partial U}\right)_V = 0$ This is typically the case.

In general Q does not exist and we can't write $\tilde{\delta} Q$ as $\tilde{\partial} Q$

However, since $\tilde{\delta} Q$ is a 1-form but not exact in 2-space $\tilde{\partial}(\tilde{\delta} Q)$ is a 2-form. This 2-form is in the complete ideal of δQ hence $\tilde{\delta} Q$ is closed.

We can use Frobenius' theorem and deduce that $\exists\ T(V, U)$ and $S(V, U)$ such that

$$\boxed{\tilde{\delta} Q = T\,\tilde{\partial} S}$$

This looks like the 2nd Law of Thermal dynamics.

With this choice the first law becomes

$$\boxed{T\tilde{\partial} S = P\tilde{\partial} V + \tilde{\partial} U}$$

§ 5.2 Maxwell and Other mathematical identities

Apply $\tilde{\partial}$ to the above equation

$$\tilde{d}(T\tilde{d}S) = \tilde{d}(P\tilde{d}V) + \underbrace{\tilde{d}\tilde{d}U}_{\to 0}$$

Assumption 1. presume $T(S,V)$ and $P(S,V)$

partial is proportional to the one form in the gradient calculation.

$$\tilde{d}T \wedge \tilde{d}S = \tilde{d}P \wedge \tilde{d}V$$

$$\left(\frac{\partial T}{\partial V}\right)_S \tilde{d}V \wedge \tilde{d}S = \left(\frac{\partial P}{\partial S}\right)_V \tilde{d}S \wedge \tilde{d}V$$

$$= -\left(\frac{\partial P}{\partial S}\right)_V \tilde{d}V \wedge \tilde{d}S$$

$$\Rightarrow \boxed{\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V}$$ One of Maxwell's indentities.

Assumption 2 $S(T,V)$ and $P(T,V)$

Sub into $\tilde{d}(T\tilde{d}S) = \tilde{d}(P\tilde{d}V)$

$$\tilde{d}T \wedge \tilde{d}S = \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V$$

$$\left(\frac{\partial S}{\partial V}\right)_T \tilde{d}T \wedge \tilde{d}V = \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V$$

$$\Rightarrow \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

Assumption : Divide eqn by T and apply $\tilde{d}$ the you can obtain

another Maxwell identity. By dividing (5.2) by T and then taking the exterior derivative we get

$$\frac{1}{T}\tilde{d}P \wedge \tilde{d}V - \frac{P}{T^2}\tilde{d}T \wedge \tilde{d}V - \frac{1}{T^2}\tilde{d}T \wedge \tilde{d}U = 0.$$

By writing $U = U(T, V)$, $P = P(T, V)$, we get

$$\frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T \wedge \tilde{d}V - \frac{P}{T^2}\tilde{d}T \wedge \tilde{d}V - \frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T \tilde{d}T \wedge \tilde{d}V = 0,$$

or

$$T\left(\frac{\partial P}{\partial T}\right)_V - P = \left(\frac{\partial U}{\partial V}\right)_T. \quad (5.6)$$

$$T\left(\frac{\partial P}{\partial T}\right)_V - P = \left(\frac{\partial U}{\partial V}\right)_T$$ assume $P(T,V)$ and $U(T,V)$

$$\boxed{T\tilde{d}S = P\tilde{d}V + \tilde{d}U}$$

$$\tilde{d}P = \left(\frac{\partial P}{\partial V}\right)_T \tilde{d}V + \left(\frac{\partial P}{\partial T}\right)_V \tilde{d}T$$

Dividing by T

$$\tilde{d}S = \frac{P}{T}\tilde{d}V + \frac{1}{T}\tilde{d}U$$

$$\frac{1}{T}\Big(\underbrace{\left(\frac{\partial U}{\partial V}\right)_T \tilde{d}V + \left(\frac{\partial U}{\partial T}\right)_V \tilde{d}T}_{\tilde{d}U}\Big)$$

$$\tilde{d}\tilde{d}S = \tilde{d}\left(\frac{P}{T}\tilde{d}V\right) + \tilde{d}\left(\frac{1}{T}\tilde{d}U\right)$$

$$0 = \left(\frac{-1}{T^2}P\tilde{d}T + \frac{1}{T}\tilde{d}(P)\right)\wedge\tilde{d}V + \left(\frac{-1}{T^2}\tilde{d}T\wedge\tilde{d}U + \underbrace{\frac{1}{T}\tilde{d}\tilde{d}U}_{\to 0?}\right)$$

$$= \frac{-P}{T^2}\tilde{d}T\wedge\tilde{d}V + \frac{1}{T}\left[\cancel{\left(\frac{\partial P}{\partial V}\right)_T\tilde{d}V}^{\,0} + \left(\frac{\partial P}{\partial T}\right)_V\tilde{d}T\right]\wedge\tilde{d}V - \frac{1}{T^2}\tilde{d}T\wedge\tilde{d}U$$

$$= -\frac{P}{T^2}\tilde{d}T\wedge\tilde{d}V + \frac{1}{T}\left(\frac{\partial P}{\partial T}\right)_V\tilde{d}T\wedge\tilde{d}V - \frac{1}{T^2}\tilde{d}T\wedge\tilde{d}U$$

$$\frac{1}{T^2}\left[\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}V + \left(\frac{\partial U}{\partial T}\right)_V\tilde{d}T\right]$$

$$\frac{1}{T^2}\left[\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}V + \cancel{\left(\frac{\partial U}{\partial T}\right)_V\tilde{d}T}^{\,0}\right]\wedge\tilde{d}T$$

$$-\frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T\tilde{d}T\wedge\tilde{d}V$$

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Phase Space in Mechanics

From section §2.3 c of The Geometry of Physics by Frankel

In classical Mechanics we describe a system using generalized coordinates,

$$q^1, \dots, q^n \quad \} \text{ compactly called } q$$

These form an n-dimensional manifold M that we call configuration space.

The lagrangian is a function of q and $\dot{q}$, where $\dot{q} = \frac{dq}{dt}$ which also has n-coordinates

These 2n coordinates $q, \dot{q}$ completely specify the state

The $\dot{q}$ are generalized velocities and are in T_pM Therefore, $(q, \dot{q})$ is in the tangent bundle, TM

The lagrangian, $L(q, \dot{q})$, is a map $L: TM \to \mathbb{R}$.

For Hamiltonian mechanics, we need the generalized momenta

$$\boxed{p_i(q, \dot{q}) \equiv \frac{\partial L}{\partial \dot{q}^i}} \to \text{one form, ie in co-tangent space}$$

↳ sub script means p is in cotangent space

To build the Hamiltonian, we need the Lagrangian and a transformation

$$(q, \dot{q}) \to (q, p)$$

This is not simply changing coordinates

To see this suppose we have a change in generalized coordinates

$$q_u \longrightarrow q_v$$

This can be described as

prime denotes new coordinates

$$q_u = q_v(q_u)$$

$$\dot{q}_v^{i'} = \left(\frac{\partial q_v^{i'}}{\partial q_u^j}\right) \dot{q}_u^j$$

Compare this with $\Lambda^{i'}_j = \frac{\partial y^{i'}}{\partial x^j}$ $\quad V^{i'} = \Lambda^{i'}_j V^j$ (contravariant) (vectors do this)

The p's transform as follows:

$$p^{v}_{i'} \equiv \frac{\partial L}{\partial \dot{q}^{i'}_{v}} = \left(\overbrace{\frac{\partial L}{\partial q^{j}_{u}} \frac{\partial q^{j}_{u}}{\partial \dot{q}^{i'}_{v}}}^{0} + \frac{\partial L}{\partial \dot{q}^{j}_{u}} \frac{\partial \dot{q}^{j}_{u}}{\partial \dot{q}^{i'}_{v}} \right)$$

$$p^{v}_{i'} = p^{u}_{j} \left(\frac{\partial \dot{q}^{j}_{u}}{\partial \dot{q}^{i'}_{v}} \right)$$ This shows this is covariant and must live in the cotangent space

Compare with $\Lambda^{j}{}_{i'} \equiv \frac{\partial x^{j}}{\partial y^{i'}}$

$\dot{q}$ is in the tangent space (vector)

p is in the cotangent space (one-form)

Hence computing p is not only changing variables but is really a map

$$p: TM \longrightarrow T^{*}M$$ right hand side could be $T_p^{*}M$ but if we add q then $T^{*}M$

$T^{*}M$ is the phase space (q,p)

The Hamiltonian is a map H s.t.

$$H: T^{*}M \longrightarrow \mathbb{R} \qquad H(q,p)$$

The Lagrangian: $L(q,\dot{q}) = T(q,\dot{q}) - U(q)$

metric

Where the KE is: $T(q,\dot{q}) = \frac{1}{2} g_{jk} \dot{q}^{j} \dot{q}^{k}$

Example: Suppose we have 2 masses in 1D

$M = \mathbb{R}^2$ and $TM = \mathbb{R}^4$

$T = \frac{1}{2} m_1 (\dot{q}_1)^2 + \frac{1}{2} m_2 (\dot{q}_2)^2$

need with $g_{ij} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$

Example: If we have a mass in 2D

$T = \frac{1}{2} m (\dot{x} + \dot{y})^2$ [cartesian]

with $g_{ij} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$

Involved q

$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$ $g_{ij} = \begin{bmatrix} m & 0 \\ 0 & r^2 \end{bmatrix}$ [polar coordinates]

In general $p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial T}{\partial \dot{q}^i} = g_{ij}(q)\dot{q}^j$

∂T can be used to define a Riemannia metric

$$\langle \dot{q}, \dot{q} \rangle = g_{ij}(q)\dot{q}^i\dot{q}^j$$

Kinetic Energy is ½ the length squared of the velocity vector

The generalized momenta p is the covariant version of the generalized velocity.

Example 1: $p_1 = m_1\dot{q}^1$ and $p_2 = m_2\dot{q}^2$

In general $p_i = g_{ij}\dot{q}^j$ and $\dot{q}^i = g^{ij}p_j$

§ 5.4 Hamiltonian Vector Fields

Given a Lagrangian, we can obtain the equations of motion from the Euler Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

Hamiltonian

$$\mathcal{H}(q,p) = p\dot{q} - L$$

Hamiltons' eqns

$$\dot{q} \, \frac{\partial \mathcal{H}}{\partial p} \quad + \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$$

Phase Space is the tangent bundle T^*M, which includes M and T_p^*M

On T^*M, which is a manifold, we define a 2-form,

$$\boxed{\tilde{\omega} = \tilde{d}q \wedge \tilde{d}p}$$ area in phase space

can be called M?

Take a curve on T^*M of the form

$$\{ q = f(t),\ p = g(t) \}$$

Which is a solution to Hamilton's equations. The tangent vector to the curve is,

$$\bar{U} = \frac{d}{dt} = \dot{f}\frac{\partial}{\partial q} + \dot{g}\frac{\partial}{\partial p}$$ basis vectors

Theorem: If $\bar{u}$ is a tangent vector to the solution curve then $\mathcal{L}_{\bar{V}}\tilde{\omega} = 0$

Proof: From a formula (4.67)

$$\mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})] + \overset{=0}{(\tilde{d}\tilde{\omega})}(\bar{u})$$

The 2^{nd} term is 0 since $\tilde{\omega}$ is a 2-form on a 2 dim Manifold

$$\Rightarrow \quad \mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\tilde{\omega}(\bar{u})]$$

$$= \tilde{d}[\tilde{d}q \wedge \tilde{d}p\,(\bar{u})]$$

$$= \tilde{d}[(\tilde{d}q \otimes \tilde{d}p - \tilde{d}p \otimes \tilde{d}q)(\bar{u})]$$

$$= \tilde{d}[(\tilde{d}q(\bar{u})\tilde{d}p - \underbrace{\tilde{d}p(\bar{u})}_{\text{coefficient of 1 form}}\overbrace{\tilde{d}q}^{\text{one form}})]$$

But $\bar{u} = f\frac{\partial}{\partial q} + g\frac{\partial}{\partial p}$ which yields

$$\tilde{d}q(\bar{u}) = \dot{f} \quad \text{and} \quad \tilde{d}p(\bar{u}) = \dot{q}$$

$$\Rightarrow \quad \mathcal{L}_{\bar{V}}\tilde{\omega} = \tilde{d}[\dot{f}\,\tilde{d}p - \dot{g}\,\tilde{d}q]$$

However $\dot{f} = \frac{\partial H}{\partial p}$ and $\dot{g} = -\frac{\partial H}{\partial q}$ from Hamiltons eqn

$$\mathcal{L}_{V}\tilde{\omega} = \tilde{d}\left[\frac{\partial H}{\partial q}\tilde{d}q + \frac{\partial H}{\partial p}\tilde{d}p\right]$$

$$= \tilde{d}[\tilde{d}H] = 0$$

The area in phase space is conserved along solns to Hamiltons equations

A Vector field with $\mathcal{L}_{\bar{V}}\tilde{\omega} = 0$ is a Hamitonian Vector field. $\bar{u}$ is tangent to the curves in phase space

The System is conservative (H is constant along Solns)

$$\mathcal{L}_{\bar{V}} H = \frac{dH}{dt} = \dot{q}\frac{\partial H}{\partial q} + \dot{p}\frac{\partial H}{\partial p}$$

$$= \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial H}{\partial p}$$

$$= 0$$

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§ 5.5 canonical transformation

p and q are not unique. P and Q are canonical if $\tilde{d}q \wedge \tilde{d}p = \tilde{d}P \wedge \tilde{d}Q$

This requires

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$$

example $Q = p$ and $P = -q$

check: $\frac{\partial p}{\partial q}\frac{\partial(-q)}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial(q)}{\partial q} = 1$

§ 5.6 Map between vectors and 1-forms by $\tilde{\omega}$

$\tilde{\omega} = \tilde{d}q \wedge \tilde{d}p$ can be used like the metric tensor to convert vectors to forms and vice versa. Suppose $\bar{V}$ is a vector field on M. then $\tilde{V} = \tilde{\omega}(\bar{V}) = \tilde{d}q \wedge \tilde{d}p\,(\bar{V})$

$$= (\tilde{d}q \otimes \tilde{d}p - \tilde{d}p \otimes \tilde{d}q)(\bar{V})$$

$$= \tilde{d}q(\bar{V})\,\tilde{d}p - \tilde{d}p(\bar{V})\,\tilde{d}q$$

If $\bar{V} = V^1 \partial/\partial q + V^2 \partial/\partial p$ then

$$\tilde{V} = V^1 \tilde{d}p - V^2 \tilde{d}q$$

when these become 1-form components doesn't this lower the indices?

$$= -V^2 \tilde{d}q + V^1 \tilde{d}p$$

We can write $(\tilde{V})_i = \omega_{ij} V^j$ and deduce that

$$\omega_{ij} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \omega^{ij} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Using ω^{ij} we can find $\bar{V}$ given $\tilde{V}$

§ 5.7 Poisson Bracket

Say f, g are functions on M and define

$$\bar{X}_f = \overline{df} \quad \text{and} \quad \bar{X}_g = \overline{dg}$$

These are the vector versions of the gradient, From above,

$$\tilde{d}f = \frac{\partial f}{\partial q}\tilde{d}q + \frac{\partial f}{\partial p}\tilde{d}p$$

then

$$\bar{X}_f = \bar{\partial} f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p}$$

$$\bar{X}_g = \bar{\partial} g = \frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}$$

We can then define the Poisson bracket

$$\begin{aligned}
\{f,g\} &\equiv \tilde{\omega}(\bar{X}_f, \bar{X}_g) \\
&= \omega_{ij} X_f^i \, X_g^j \\
&= (X_f)_j \, X_g^j \\
&= \tilde{\partial} f(\bar{X}_g) = \langle \tilde{\partial} f, \bar{X}_g \rangle
\end{aligned}$$

To evaluate this we get,

$$\{f,g\} = \left(\frac{\partial f}{\partial q}\tilde{d}q + \frac{\partial f}{\partial p}\tilde{d}p\right)\left(\frac{\partial g}{\partial p}\frac{\partial}{\partial q} - \frac{\partial g}{\partial q}\frac{\partial}{\partial p}\right)$$

$$\boxed{\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}}$$ poisson bracket

Aside $\tilde{\partial} q\left(\frac{\partial}{\partial q}\right) = 1$

$\tilde{\partial} q\left(\frac{\partial}{\partial p}\right) = 0$

The above expression is in terms of coordinates. The expression independent of coordinates is

$$\{f,g\} = \tilde{\omega}(\bar{\partial} f, \bar{\partial} g)$$

§ 5.8 Many particle systems: symplectic forms

In 3D with no constraints, and N particles, we have 6N dim'l phase space.

The phase space in general can be said to be 2N where N is the number of generalized coordinates

then

$$\boxed{\tilde{\omega} = \sum_{A=1}^{n} \tilde{d}q^A \wedge \tilde{d}p_A}$$ Symplectic Form

The Phase space is a Symplectic Manifold.

§ 5.9 Linear Dynamical Systems: the symplectic inner product and conserval quantities

to begin consider the following hamiltonian

$$H = \frac{1}{2} \sum_{A,B=1}^{n} T^{AB} p_A p_B + V_{AB} q^A q^B$$

where we assume T^{AB} and V_{AB} are symmetric. If not, we use the fact the product of the asymetric part and a symmetric function is 0.

For symplicity, assume T^{AB} and V_{AB} are constant.

Hamilton's equations

$$\frac{dP_A}{dt} = -\frac{\partial H}{\partial q^A} = -\sum_B V_{AB} q^B$$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p_A} = \sum_B T^{AB} p_B$$

Check: $\frac{\partial H}{\partial q^c} = \frac{\partial}{\partial q^c}\left(\frac{1}{2} \sum_{A,B=1}^{n} V_{AB} q^A q^B \right)$

$$= \frac{1}{2} \sum_{A,B} V_{AB} \delta^A_c q^B + \frac{1}{2} \sum_{A,B} V_{AB} q^A \delta^B_c$$

$$= \frac{1}{2} \sum_B V_{cB} q^B + \frac{1}{2} \sum_A \underset{V_{cA},\ A \to B}{\underset{\parallel}{V_{Ac}}} q^A$$

$$= \sum_B V_{cB} q^B$$

If $\bar{Y}_{(1)}$ is a vector with components $\{ q_{(1)}^A, p_{(1)A}, A=1\ldots n \}$ and $\bar{Y}_2$ is a vector w components $\{ q^A_{(2)}, p_{(2)A}, A=1,\ldots,n \}$

then their symplectic product is

$$\tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \sum_A q^A_{(1)} p_{(2)A} - q^A_{(2)} p_{(1)A}$$

If $\bar{Y}_{(1)}$ and $\bar{Y}_{(2)}$ are both solutions, then the symplectic inner product is independent of time

$$\frac{d}{dt} \tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \frac{d}{dt} \left[\sum_{A=1}^{n} q^A_{(1)} p_{(2)A} - q^A_{(2)} p_{(1)A} \right]$$

$$= \sum_{A=1}^{n} \left\{ \frac{dq^A_{(1)}}{dt} p_{(2)A} + q^A_{(1)} \frac{dp_{(2)A}}{dt} - \frac{dq^A_{(2)}}{dt} p_{(1)A} - q^A_{(2)} \frac{dp_{(1)A}}{dt} \right\}$$

Using Hamilton's eqns next,

$$\frac{d}{dt} \tilde{\omega}(\bar{Y}_{(1)}, \bar{Y}_{(2)}) = \sum_{A,B}^{n} \{ T^{AB} P_{(1)B} P_{(2)A} - V_{AB} q_{(1)}^{A} q_{(2)}^{B} - T^{AB} P_{(2)B} P_{(1)A} + V_{AB} q_{(2)}^{B} q_{(1)}^{A} \} = 0$$

If T^{AB} and V_{AB} are independent time then it follow that if $\bar{Y}_{(1)}$ is a soln then so is $\frac{d\bar{Y}_{(1)}}{dt}$

This motivates defining the canonical energy as $E_c(\bar{Y}) = \tilde{\omega}(\frac{d\bar{Y}}{dt}, \bar{Y})$

It can be determined that $E_c(\bar{Y}) = \mathcal{H}$ evaluated at $\bar{Y}$

$$E_c(\bar{Y}) = \tfrac{1}{2}\tilde{\omega}(\dot{\bar{Y}}, \bar{Y}) = \sum_{A}^{n} (\dot{q}_{(1)}^{A} P_{(2)A} - q_{(2)}^{A} \dot{P}_{(1)A})$$

$$= \frac{1}{2} \sum_{A,B} T^{AB} P_{(1)B} P_{(2)A} + V_{AB} q^{B}_{(1)} q^{A}_{(2)} = \mathcal{H}$$

So the independence of $\mathcal{H}$ w.r.t yields the conservation of $\mathcal{H}$ or the total Mechanical Energy

Other conserved quantities

In general, T^{AB} and V_{AB} can depend on the coordinates $\{x^i\}$.

If $\exists\, \bar{u}$ such that $\mathcal{L}_{\bar{u}} T^{AB} = 0 = \mathcal{L}_{\bar{u}} V_{AB}$ then there are conserved quantities associated to $\bar{u}$

This can yield expressions for Linear Momentum or angular moment conservation

Noethers theorem?

Exam Content

Lec 23 - Nov 30th

§5.11 Rewritting Maxwell's equations in differential forms
§C electromagnetism

We can non-dimensionalize maxwell's equations in such away that $c = \mu_0 = \epsilon_0 = 1$ to get

$$A \quad \vec{\nabla} \times \vec{B} - \frac{\partial E}{\partial t} = 4\pi \vec{J} \qquad \text{Amperes Law}$$

electric field, magnetic field

$$B \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \text{Faraday's Law}$$

$$C \quad \vec{\nabla} \cdot \vec{B} = 0$$

$$D \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

Gauss

charge

We will rewrite these using a metric and $\tilde{d}$. The relativistic invariant form requires the Faraday 2-form

$$F_{\mu\nu} = \begin{array}{c|cccc} & t & x & y & \nu = z \\ \hline \mu = t & 0 & -E_x & -E_y & -E_z \\ x & E_x & 0 & B_z & -B_y \\ y & E_y & -B_z & 0 & B_x \\ z & E_z & B_y & -B_x & 0 \end{array} \qquad \mu, \nu = t, x, y, z$$

rows columns

→ How is this a 2-Form

what does this look like?

Then $\tilde{d}F$ is a 3-form on a 4D manifold. Since $\tilde{d}F$ is a 3-form on a 4D Manifold, there are C^4_3 different equations, $C^4_3 = 4$

We can write $F = F_{\mu\nu}\, \tilde{d}\mu \wedge \tilde{d}\nu$, we compute,

$$\tilde{d}P = F_{\mu\nu,\gamma}\, \tilde{d}\gamma \wedge \tilde{d}\mu \wedge \tilde{d}\nu$$

It is observed that $\tilde{d}F = 0$ iff $F_{[\mu\nu,\gamma]} = 0$

① $F_{[xy,z]} = F_{xy,z} + F_{yz,x} + F_{zx,y} = 0$

$B_{z,z} + B_{x,x} \;\; B_{y,y} = 0$ Div of Magnetic field = 0

or $\vec{\nabla} \cdot \vec{B} = 0$ Eqn C

② $F_{[xy,t]} = F_{xy,t} + F_{yt,x} + F_{tx,y} = 0$

$B_{z,t} + E_{y,x} - E_{x,y} = 0$ the z equation of eqn B

③ $F_{[yz,t]} = F_{yz,t} + F_{zt,y} + F_{ty,z} = 0$

$= B_{x,t} + E_{z,y} - E_{y,z} = 0$ the x eqn in B

④ $F_{[xz,t]} = F_{xz,t} + F_{zt,x} + F_{tx,z} = 0$

$-B_{y,t} + E_{z,x} - E_{x,z} = 0$ the y eqn of B

For the other equations we need the special relativistic metric

$$(g_{\mu\nu}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \text{Lorentzian metric?}$$

This allows us to find the 2-vector $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} = g^{\mu\alpha} F_{\alpha\beta} g^{\beta\nu}$

Note: $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}$$

The final for eqn's are. $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ where $J^t = \rho$, $J^i = (J)^i$ $i = x, y, z$

We check 4 different equations

① $F^{t\nu}{}_{,\nu} = F^{tx}{}_{,x} + F^{ty}{}_{,y} + F^{tz}{}_{,z} = 4\pi J^t$

$= E_{x,x} + E_{y,y} + E_{z,z} = 4\pi\rho$ eqn D

② $F^{x\nu}{}_{,\nu} = F^{xt}{}_{,t} + F^{xy}{}_{,y} + F^{xz}{}_{,z} = 4\pi J^x$

$= -E_{x,t} + B_{z,y} - B_{y,z} = 4\pi J^x$ This is the x eqn in A

③ $F^{y\nu}{}_{,\nu} = F^{yt}{}_{,t} + F^{yx}{}_{,x} + F^{yz}{}_{,z} = 4\pi J^y$

$= -E_{y,t} - B_{z,x} \quad B_{x,z} = 4\pi J^y$ The y eqn in A

④ $F^{z\nu}{}_{,\nu} = F^{zt}{}_{,t} + F^{zx}{}_{,x} + F^{zy}{}_{,y} = 4\pi J^z$

$= -E_{z,t} + B_{y,x} - B_{x,y} = 4\pi J^z$ The z eqn in A.

Observe that $\tilde{d}F$ is coordinate independent, However $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ is coordinate dependent.

Given a basis of the tangent space $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ then, we can define the volume 4-form as

$$\tilde{\omega} = \tilde{d}t \wedge \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$$

We define $*\tilde{F} = \frac{1}{2}\tilde{\omega}(F)$ or $(*\tilde{F})_{\mu\nu} = \frac{1}{2}\omega_{\alpha\beta\mu\nu}F^{\alpha\beta}$ dual or Hodge Star

Next we determine the components of this,

① $(*\tilde{F})_{tx} = \frac{1}{2}\omega_{\alpha\beta tx}F^{\alpha\beta}$

$= \frac{1}{2}\omega_{yztx}F^{yz} + \underbrace{\frac{1}{2}\omega_{zytx}F^{zy}}_{\text{antisymmetric}}$

$yztx \to -ytzx \to ytxz \to -tyxz \to txyz$ $+1$???

$(*\tilde{F})_{tx} = B_x$

② $(*\tilde{F})_{ty} = \frac{1}{2}\omega_{\alpha\beta ty}F^{\alpha\beta} = \frac{1}{2}(\omega_{zxty}F^{zx} + \omega_{xzty}\underbrace{F^{xz}}_{=-F^{zx}})$

$zxty \to -xzty \to xtzy \to -txzy \to txyz$ $+1$ coefficent.

$(*\tilde{F})_{ty} = B_y$

③ $(*\tilde{F})_{tz} = B_z$

④ $(*\tilde{F})_{xy} = E_z$

⑤ $(*\tilde{F})_{xz} = -E_y$

⑥ $(*\tilde{F})_{yz} = E_x$

$$(*\tilde{F})_{\mu\gamma} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & F_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & E_x & 0 \end{pmatrix}$$

The exterior derivative of this is

$$\tilde{d}(*\tilde{F}) = (*\tilde{F})_{\mu\nu,\gamma}\, \tilde{d}\gamma \wedge \tilde{d}\mu \wedge \tilde{d}\nu$$

we define $*\tilde{J} = \tilde{\omega}(\bar{J})$ and apply $\tilde{\omega}$ to the eqn $F^{\mu\nu}{}_{,\gamma} = 4\pi J^{\mu}$

$$\tilde{\omega}(F^{\mu\nu}{}_{,\gamma}) = \tilde{\omega}(4\pi J^{\mu})$$

we can write this as

$$\boxed{\tilde{d}(*\tilde{F}) = 4\pi *\tilde{J}}$$

and

$$\tilde{d}F = 0$$

§ 5.13 Vector Potential

If $\tilde{d}F = 0$ then F is closed and since it is a 2-form, $\exists$ a 1-form $\tilde{A}$ such that

$$F = \tilde{d}\tilde{A} \qquad \text{at least locally}$$

$\tilde{A}$ is the vector potential

Lec 24 Dec 5^{th}

D Dynamics of a perfect fluid

§ 5.15 Role of Lie derivatives

A perfect fluid (idealized) is one that conserves certain propreties

1. Mass
2. Entropy
3. Vorticity [will explain] $\nabla \times$ velocity

Today we will express the equations of a fluid using exterior calculus.

§ 5.16 the Comoving time derivative

The conservation of mass (continuity eqn) is:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

things convering, denisty increases

On an assigment we found

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{v}}\right)(\underbrace{\rho \tilde{\omega}}_{\text{mass}}) = 0$$

This is being written in terms of differential forms

where $\tilde{\omega} = \tilde{d}x \wedge \tilde{d}y \wedge \tilde{d}z$

The operator $\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{v}}\right)$ computes the total rate of change following the flow.

Consider the motion of a fluid parcel. If the change happens over a short time,

$$dt \ll 1$$

→ makes the following approximation valid

Then the motion is from

$$(x, y, z, t) \rightarrow (x + V^x dt,\ y + V^y dt,\ z + V^z dt,\ t + dt)$$

The difference between the two is

$$(V^x, V^y, V^z, 1)\, dt = \bar{U}$$ is a 4-vector

In the formulation, the total rate of change following the flow:

$$\mathcal{L}_{\bar{u}} \bar{W} = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) \bar{W}$$

Space time version.

where $\bar{W}$ is a 4-vector and it would need to be decomposed on the RHS

§ 5.17 Eqns of Motion

A perfect fluid conserved entropy. If S is the entropy, then the eqn is

1 of 3

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) S = 0}$$

Thermodynamics

The conservation of Linear momentum (Newton's 2nd Law) can be written as:

$$\underbrace{\frac{\partial}{\partial t} V^i + V^j \frac{\partial}{\partial x^j} V^i}_{\text{total rate of change}} + \frac{1}{\rho} \frac{\partial}{\partial x^i} P + \frac{\partial}{\partial x^i} \Phi = 0$$

F = ma ?

pressure

gravity

P is pressure

Φ is the gravitational potential

V^i is the Velocity

This eqn is a mess as we have both superscripts and subscripts added to each other. Bad!

$\frac{\partial V^i}{\partial x^j}$ is not a $\binom{1}{1}$ tensor.

→ Partial derivatives don't transform as vectors

→ How does a metric relate to a distance fcn.

Assume we have a metric

This allows us to convert the vector $\bar{V}$ to the one-form $\tilde{V}$

$$\boxed{\tilde{V} = g_1(\bar{V}, *)}$$ yeilds V_i

↳ star means empty in this case

To rewrite the non-linear term, we need the operator

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) \tilde{V}}$$

To find out what this term looks like, consider eqn 3.14 in textbook

$$(\mathcal{L}_{\bar{V}} \tilde{V})_i = V^j \frac{\partial}{\partial x^j} V_i + V_j \frac{\partial}{\partial x^i} V^j$$

Can show using $\vec{\nabla} \cdot V = 0$

$$= V^j \frac{\partial}{\partial x^j} V_i + \frac{1}{2} \frac{\partial}{\partial x^i} (V_j V^j)$$

$$\Rightarrow V^j \frac{\partial}{\partial x^j} V_i = (\mathcal{L}_{\bar{V}} \tilde{V})_i - \frac{\partial}{\partial x^i} \left(\frac{1}{2} V^2\right) \quad \text{if} \quad V^2 = \tilde{V}(\bar{V})$$

Our momentum equation in coordinate independent form becomes

2 of 3

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) \tilde{V} + \frac{1}{\rho} \tilde{d} p + \tilde{d}\left(\Phi - \frac{1}{2} V^2\right) = 0}$$

Almost like bernoulli's eqn. ??

§ 5.18 Conservation of Vorticity

The vorticity of a fluid with velocity $\bar{V}$ is

$$\vec{\nabla} \times \vec{V} \quad \text{curl of velocity}$$

we have seen that this can be written as

$$* \tilde{d} \tilde{V} \qquad \text{curl} \rightarrow 1 \text{ vector}$$

$$\tilde{d} \tilde{V} \qquad \text{curl} \rightarrow 2\text{-form}$$

To get the vorticity equation we apply $\tilde{d}$ to the equation

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) \tilde{d} \tilde{V} = \frac{1}{\rho^2} \tilde{d} \rho \wedge \tilde{d} P \qquad *$$

Case ① $p = d(\rho)$ then $\tilde{d} \rho \wedge \tilde{d} P = 0$

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\bar{V}}\right) \tilde{d} \tilde{V} = 0}$$

Vorticity is conserved following the flow

Case 2 : $p = p(\rho, s)$ then $\tilde{d}\rho \wedge \tilde{d}P \neq 0$ but

$$\boxed{\tilde{d}S \wedge \tilde{d}\rho \wedge \tilde{d}P = 0}$$

Manifold is 2D (s, ρ)

Apply $\tilde{d}$ to our equation 1 of 3

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right) \tilde{d}S = 0}$$

If we take $\tilde{d}S \wedge$ the vorticity eqn * then

$$\tilde{d}S \wedge \left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right) \tilde{d}\tilde{V} = \frac{1}{\rho^2} \, \underbrace{\tilde{d}S \wedge \tilde{d}\rho \wedge \tilde{d}p}_{\to 0}$$

or

$$\boxed{\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right) \tilde{d}S \wedge \tilde{d}\tilde{V} = 0}$$

Ertels thereom

$\tilde{d}S \wedge \tilde{d}V$ is a 3-form. By Mass conservation $\rho\tilde{\omega}$ is conserved these are both 3-forms and on a 3D manifold must be Linearly relates

$$\boxed{\tilde{d}S \wedge \tilde{d}\tilde{V} = \alpha \rho \tilde{\omega}}$$

α is a function and must exist!

Since $\tilde{d}S \wedge \tilde{d}\tilde{V}$ and $\rho\tilde{\omega}$ is consered it follows that α is consered.

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\vec{v}}\right)\alpha = 0$$

claim: $\alpha = \frac{1}{\rho} \vec{\nabla}S \cdot \vec{\nabla} \times \vec{V}$

proof: take the dual of

$$\tilde{d}S \wedge \tilde{d}V = \alpha \rho \tilde{\omega}$$

$$*(\tilde{d}S \wedge \tilde{d}V) = *(\alpha\rho\tilde{\omega})$$

Note: $\tilde{d}S \wedge \tilde{d}V = S_{,i}\, \tilde{d}x^i \wedge \epsilon^{ijk} V_{k,j}\, \tilde{d}x^j \wedge \tilde{d}x^k$

$$= \epsilon^{ijk} S_{,i} V_{k,j}\, \tilde{d}x^i \wedge \tilde{d}x^j \wedge \tilde{d}x^k$$

Comparing the coefficent of $\tilde{\omega}$ we get.

$$\alpha = \frac{1}{\rho} \vec{\nabla} S \cdot \vec{\nabla} \times \vec{V}$$ Ertel Potential Vorticity.